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# English summary

The Navier-Stokes equations, the Euler equations, the equations of elasticity and expressions derived from those, are in most cases treated in Cartesian coordinates. In some cases it can necessary to handle those equations in other coordinate systems. In the cases of cylindrical coordinates, for example in the description of the flow around acoustic antennas, it is natural to use cylinder coordinates. In this report, we present the formalism necessary to handle the mentioned equations and related expressions in generalized coordinates. The formalism include tensor analysis, developed during 1850-1900 by Gregorio Ricci Kurbastro, Tullio Levi-Civita, Sophus Lie and others. Albert Einstein used tensor analysis as the mathematical basis for the General Theory of Relativity. In this report we will limit our self to describe the classical fluid equations in generalized coordinates.

The tensor-theory can appear to be difficult and one can ask if it is necessary to go through all these complicated calculations. Can't they be found at the web or in standard collections of formulas? We have looked for expressions, for example $\nabla \cdot (\nabla(\rho \mathbf{T}))$, where $\mathbf{T}$ is the momentum flux density tensor that appears in Lighthill's equation. We could not find this derived in cylinder coordinates and it was necessary to calculate it by hand to achieve our goals. In the analysis of flow around an acoustic antenna, various tensors appear, for example the strain rate tensor, structural tensors and tensorial expressions involved in the RANS equations, it was necessary to follow the formalism of tensor analysis in detail.

With data given in cylinder coordinates, it is natural to do the analysis also in cylinder coordinates. Physical components of both vectors and tensors are used in the physical interpretations of the data.

Although the treatment in cylinder coordinates addressed in this report only is directly applicable to a limited number of applications, the concept of tensor analysis is fundamental in practically all applications of continuum mechanics.

# Sammendrag

Navier-Stokes ligninger, Euler ligningene og elastisitets ligningene og uttrykk avledet av disse håndteres oftest i Cartesiske koordinater. Det kan likevel være avgjørende å kunne håndtere disse ligningene i andre koordinatsystemer, for eksempel i sylinder-koordinater i tilfellet strømning omkring eller i cylindriske rør, for eksempel strømning rundt akustiske antenner. I dene rapporten presenterer vi formalismen som må til for å uttrykke de nevnte ligningene i generaliserte koordinater. Formalismen omfatter tensoranalyse, som ble utviklet i tidsrommet 1850-1900 av Gregorio Ricci Kurbastro, Tullio Levi-Civita, Sophus Lie og andre. Albert Einstein benyttet tensor analysen som som matematisk fundament for generell relativitetsteori. I denne rapporten vil vi begrense oss til å beskrive de klassiske fluid ligningene i generaliserte koordinater.

Tensor-teorien kan virke tung og vanskelig og en kan spørre seg om det er nødvendig å gjennomgå alle disse kompliserte regningene, at det ikke bare er å søke på “webben” eller i en standard formesamling etter nødvendige uttrykk. Vi har lett etter uttrykk som $\nabla \cdot (\nabla(\rho \mathbf{T}))$, her er $\mathbf{T}$ er momentum-fluks-tetthets-tensoren. Uttrykket forekommer i Lighthill’s ligning. Vi fant ikke dette uttrykket i sylinderkoordinater og for å nå målet var det nødvendig å følge tensor analysens prosedyrer til punkt og prikke. I analysen av strøning omkring en akustisk antenne så inngår flere tensorer som for eksempel deformasjonsrate tensoren, struktur tensorer og tensorielle uttrykk som forekommer i RANS ligningene. Med data gitt i sylinder koordinater er det naturlig å gjennomføre analysen i sylinder koordinater. Fysikalske komponenter beregnes både for vektor og tensor under den fysiske tolkningen av dataene.

Selv om behandlingen av cylinder koordianter i denne rapporten kun har begrenset anvendelse, er konseptene i tensor analyse av fundamental betydning i praktisk talt alle anvendelser av kontinumsmekanikk.

# Contents

# 1 Introduction

In an effort to simulate the sound excited by a turbulent boundary layer surrounding a seismic streamer we encounter an in-homogeneous wave equation called Lighthill's equation. This equation is a result of re-writing the Navier Stokes equations for compressible flows without making any physical simplifications. The source terms contained in Lighthill's equation, which are of importance for example for turbulent flows, are the cause of sound propagating from the turbulent region and into the surroundings. They may be classified as a quadrupole source. Replacing the turbulent source by quadrupoles is called Lighthill's analogy. Lighthill's theory has had huge impact in the field of aero-acoustics and aero-elasticity. For details on Lighthill's analogy, see [11, 9, 10]. The inhomogeneous wave equation written in the form suggested by Lighthill is

$$\frac{1}{C^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \nabla \cdot \nabla \cdot (\rho \mathbf{v}\mathbf{v} + \boldsymbol{\sigma}). \tag{1.1}$$

Here $p$ is the pressure, $C$ the local sound speed, $\rho$ the density, $\mathbf{v}$ is the velocity and $\boldsymbol{\sigma}$ is a stress tensor caused by thermal and viscous dissipation. The expression

$$\mathbf{T} = \rho \mathbf{v}\mathbf{v} + \boldsymbol{\sigma}, \tag{1.2}$$

is a second rank tensor and

$$\nabla \cdot (\nabla \cdot \mathbf{T}),$$

the *double divergence* of a tensor is a *scalar* which is a zero rank tensor.

In our work on seismic streamers, we have learned that noise caused by the impact from the external turbulent boundary layer, called flow noise, can reduce the quality of the data sampled by those systems. To better understand the nature of the noise and its impact, we have used as input, the data from a simulation of a turbulent boundary layer into equation 1.1 to simulate the noise in the streamer. The streamer is shaped as a cylinder and it has been convenient to use cylinder coordinates in the simulations. We have not been able to find terms like $\nabla \cdot (\nabla \cdot \mathbf{T})$ written out in cylinder coordinates, neither in the literature nor on the web. It has been necessary to calculate these terms the hard way by hand following the recipe given in this report. It has not been a waste to prepare this report since we also encounter several other tensorial expression that enter into our analysis of turbulent flows surrounding acoustic antennas and that we need to be able to fully control.

There is also a section devoted to the kinematics of rotating coordinate systems, and the equations of elasticity. The tensor analysis as presented in this report is based on the general treatment of Heinbockel, Irgens and Lovelock and Rund, see [3], [5] and [12]. Tensor analysis is also a basic ingredient in differential geometry. An introduction to tensor analysis and differential geometry is given in Kreyszig's book, see [7].

## 2 Basic terminology

Vectors and tensors discussed in this report are usually applied in Euclidian space $E_3$ also denoted $\mathbf{R}^3$, but the theory presented can to some extent be extended to n-dimensional differentiable manifolds $X_n$ equipped with an affine connection, see [12] chapter 3. An intermediate step is the Riemann manifold $V_n$ which is at least equipped with a metric $g_{ij}$ from which distances in space, lengths of and angle between vectors can be calculated.

### 2.1 Some conventions

Considering two coordinate systems in which a point $\mathcal{P}$ has coordinates $x^1, \ldots, x^n$ and $\overline{x}^1, \ldots, \overline{x}^n$. These two n-tuples are related through the transformations

$$\begin{aligned}\overline{x}^1 &= \overline{x}^1(x^1, \ldots, x^n),\\ &\ldots\\ \overline{x}^n &= \overline{x}^n(x^1, \ldots, x^n).\end{aligned}$$

For convenience the n-tuple $x^1, \ldots, x^n$ is denoted by $x^i$, and the transformations above are simply written $\overline{x}^i = \overline{x}^i(x^j)$.

We assume the Einstein summation convention. An index expressed by lower case Latin letters $i, j, k, \ldots$ occurring twice implies summation. For example $a^i_i = \Sigma^n_{i=1} a^i_i$.

The Kronecker delta is

$$\delta^j_i = \begin{cases} 0, & \text{if} \quad i \neq j \\ 1, & \text{if} \quad i = j. \end{cases}$$

### 2.2 Material/Lagrangian and spatial/Eulerian coordinates

Two reference systems of special relevance in mechanics are the *Lagrangian* that also is called the *material* reference system, and the *Eulerian* that also is called the *spatial* reference system. A detailed discussion of these systems is given in ([1]). The terminology most commonly used in fluid mechanics for reference systems is Lagrangian and Eulerian while the more physically intuitive expressions “material” and “spatial” are not so much in use.

The material reference system is connected to material particles. The material coordinates of a particle do not change in time during motion. The material coordinates of a particular particle can be viewed as that particles label. They are locked to that particle as time evolves. On the other hand, the spatial coordinates are not linked to any particular particle. They are locked to a position in a particular spatial reference system. Let $\overline{x}^i$ be the material coordinate of a given particle. That particle must also have a spatial coordinate $x^j = x^j(\overline{x}^i)$. Since at one instant of time there can be only one particle in a particular point of space and a particle can only be in only one spatial point there must be a one to one correspondence between the material and the spatial coordinates for that particle. The function $x^j(\overline{x}^i)$ is bijective. The use of material $\overline{x}^i$ or spatial $x^i$ coordinates must be equivalent. The coordinate transforms given by

$$\overline{x}^j = \overline{x}^j(x^i), \quad i,j = 1,\ldots,n \tag{2.1}$$

with inverse

$$x^i = x^i(\overline{x}^j), \tag{2.2}$$

are bijective and $C^\infty$. The manifold on which they are defined is an *n-dimensional differentiable manifold*. It is denoted $X_n$.

Through this we can assure that the mechanics of particles expressed in the material and spatial reference systems can be assessed through the formalism of tensor calculus. The theory presented under is applicable in a very wide framework and the material/spatial reference systems covers a very special but anyway relevant case given here as an example.

Consider the coordinates $x^i$ and $\overline{x}^i$, both assigned to a point $\mathcal{P}$ on a differentiable manifold $X_n$ and satisfying the mappings given by (2.1) and (2.2). The Kronecker delta can be written

$$\delta^i_j = \frac{\partial x^i}{\partial \overline{x}^s}\frac{\partial \overline{x}^s}{\partial x^j} \tag{2.3}$$

and

$$\delta^k_l = \frac{\partial \overline{x}^k}{\partial x^s}\frac{\partial x^s}{\partial \overline{x}^l}. \tag{2.4}$$

The Jacobian of the transformation $x^i = x^i(\overline{x}^j)$ is

$$J = \frac{\partial(x^1,\ldots,x^n)}{\partial(\overline{x}^1,\ldots,\overline{x}^n)}. \tag{2.5}$$

Using the product rule for determinants we get

$$\frac{\partial(x^1,\ldots,x^n)}{\partial(\overline{x}^1,\ldots,\overline{x}^n)} \cdot \frac{\partial(\overline{x}^1,\ldots,\overline{x}^n)}{\partial(x^1,\ldots,x^n)} = J \cdot J^{-1} = 1,$$

so neither $J$ nor $J^{-1}$ can be zero. This must always be assured when selecting reference frames.

*Scalars, vectors* and *tensors* are all familiar expressions that most of us encounter without any deeper reflections and concerns. By a scalar field we think of a single valued function that varies through space and time. By vectors and tensors we associate certain collections of numbers. In fact, these entities carry a deeper meaning which has proved very useful to express mechanical quantities and the relations between them.

Let $x^i$ and $\overline{x}^i$ represent coordinates of two reference systems satisfying with (2.1) and (2.2) that fulfill the requirements stated above. Consider two single valued functions $s$ in the $x^i$ system and $\overline{s}$ in the $\overline{x}^i$ system. Let the point P have the coordinates $x^i$ and $\overline{x}^i$. We say that $s$

is a *scalar* if $\overline{s}(\overline{x}^i) = s(x^i)$ in point P. A scalar is and *invariant*. If this is satisfied not only in a particular point but for all points in $X_n$, we say that $s$ represents a scalar field. A scalar field is independent on reference system, a very convenient behavior when utilized in the description of invariant physical fields. An example of a single valued function that is not a scalar field is any of the components of a vector field. They depend on reference system, but the vector is an invariant.

A scalar field is denoted a tensor field of rank or order zero.. Vectors and tensor fields, are natural extensions of the scalar field to higher rank. A vector $\mathbf{v}$, with components $v^i$, is considered a first rank tensor. The components $v^i$ specify $\mathbf{v}$ in the $x^i$ system while $\overline{v}^i$ specify its components in the $\overline{x}^i$ system. Both component sets refer to the same object $\mathbf{v}$. For that to be the fulfilled, certain transformation laws must be satisfied for the components (as will be discussed later). These arguments can be extended to tensors of higher rank.

## 2.3 Relative scalars, vectors and tensors

Let $x^i$ and $\overline{x}^i$ be the coordinates assigned to a point $P$ on a differentiable manifold $X_n$. A function $s(x^i)$ on $X_n$ is a relative scalar of weight $W$ if it transforms as

$$\overline{s}(\overline{x}^i) = J^W s(x^j). \tag{2.6}$$

If $W = 0$, $s$ is called a scalar (as we have seen), also called an absolute scalar. If $W = 1$ and $s$ satisfy (2.6), it is called a scalar density. For example $g = \sqrt{|\det(g_{ij})|}$ where $g_{ij}$ is the metric tensor, is a scalar density.

A tuple $A^i$ that transforms as

$$\overline{A}^i(\overline{x}^k) = J^W \frac{\partial \overline{x}^i}{\partial x^j} A^j(x^l) \tag{2.7}$$

is called a contravariant vector of weight $W$. If $W = 0$ it is called an absolute contravariant vector or simply a contravariant vector. A tuple $A_i$ is called a covariant vector of weight $W$ if it transforms as follows

$$\overline{A}_i(\overline{x}^k) = J^W \frac{\partial x^j}{\partial \overline{x}^i} A_j(x^l). \tag{2.8}$$

Here we have used both sub and superscripts. Their meaning become clear when we express vectors in relation to contravariant and covariant vector bases. An example of a contravariant vector is the tangent vector (velocity vector) $v^i$. We have

$$v^i = \frac{dx^i}{ds}. \tag{2.9}$$

According to the chain rule,

$$\overline{v}^i = \frac{d\overline{x}^i}{ds} = \frac{\partial \overline{x}^i}{\partial x^j}\frac{dx^j}{ds} = \frac{\partial \overline{x}^i}{\partial x^j} v^j,$$

which shows that the tangent vector $v^i$ is a contravariant vector. On the other hand consider the scalar field $\psi(x^i)$. Again applying the chain rule, the components of the gradient becomes

$$\frac{\partial\overline{\psi}(\overline{x}^i)}{\partial\overline{x}^j} = \frac{\partial\psi(\overline{x}^i)}{\partial\overline{x}^j} = \frac{\partial x^k}{\partial\overline{x}^j}\frac{\partial\phi(x^i)}{\partial x^k}, \tag{2.10}$$

showing that the gradient of a scalar is a covariant vector.

Relative tensors are defined in the same way. A second rank relative tensor is a contravariant tensor of weight $W$ if it transforms as follows

$$\overline{A}^{ij} = J^W \frac{\partial\overline{x}^i}{\partial x^k}\frac{\partial\overline{x}^j}{\partial x^l}A^{kl}. \tag{2.11}$$

For $W = 0$, $A^{ij}$ is called a contravariant tensor of rank 2.

A mixed relative type $(1,1)$ tensor transforms as

$$\overline{A}^i{}_j = J^W \frac{\partial\overline{x}^i}{\partial x^k}\frac{\partial x^l}{\partial\overline{x}^j}A^k{}_l. \tag{2.12}$$

It is now easy to define a type $(r,s)$ tensor density

$$\overline{A}^{j_1\ldots j_r}{}_{k_1\ldots k_s} = J^W \frac{\partial\overline{x}^{j_1}}{\partial x^{l_1}}\cdots\frac{\partial\overline{x}^{j_r}}{\partial x^{l_r}}\frac{\partial x^{m_1}}{\partial\overline{x}^{k_1}}\cdots\frac{\partial x^{m_s}}{\partial\overline{x}^{k_s}}A^{l_1\ldots l_r}{}_{m_1\ldots m_r}. \tag{2.13}$$

Relative tensors of weight $W = 1$ are generally called tensor densities while relative tensors of weight $W = 0$ are called absolute tensors. For simplicity they are just called tensors.

# 3 Base vectors

In this section we consider vectors and tensors on a differentiable manifold equipped with a metric. A vector is represented by its components $A_i$ or $A^i$, but sometimes it is given without any explicit reference to the coordinates as $\mathbf{A}$. We say that it is given on coordinate free form. The components $A_i$ or $A^j$ of $\mathbf{A}$ express the vector using appropriate base vectors. In a Riemann space $V_n$, one set of base vectors are tangents to the coordinate lines, we call them covariant base vectors. They are written $\{\mathbf{g}_i\}$. There is also a set of reciprocal base vectors $\{\mathbf{g}^i\}$ which are normals to the coordinate surfaces $\{\mathbf{g}^i\}$, (see [3]). We have

$$\mathbf{g}_i \cdot \mathbf{g}^k = \delta_i^k.$$

In Cartesian coordinate we have the orthonormal base $\{\mathbf{e}_i\}$. $\mathbf{g}_i$ or $\mathbf{g}^i$ can be expressed as linear combinations of the Cartesian base vectors as

$$\mathbf{g}_i = a_{ij}\mathbf{e}_j \quad \text{and} \quad \mathbf{g}^i = a^{ij}\mathbf{e}_j$$

where $a_{ij}$ and $a^{ij}$ are matrices.

A vector $\mathbf{A}$ can be expressed in Cartesian coordinates by the components $a_i$ as

$$\mathbf{A} = a_i \mathbf{e}_i. \tag{3.1}$$

A vector $\mathbf{A}$ can also be expressed as linear combinations of the *co* and *contravariant* bases $\{\mathbf{g}_i\}$ and $\{\mathbf{g}^i\}$. The covariant base vectors $\mathbf{g}_i$ are defined as tangents to the coordinate lines $\mathbf{r}(a^1, \ldots, x^i, \ldots, a^n)$, where $a^1, \ldots, a^n$ are constants. For example for n=3, the coordinate lines are the family of curves $\mathbf{r}(x^1, a^2, a^3)$, $\mathbf{r}(a^1, x^2, a^3)$ and $\mathbf{r}(a^1, a^2, x^3)$ where $a^1, a^2, a^3$ are constants. $\mathbf{g}_i$ is calculated as follows

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial x^i}. \tag{3.2}$$

The base vectors $\mathbf{g}_i$ obey a covariant like transformation

$$\overline{\mathbf{g}_i} = \frac{\partial \mathbf{r}}{\partial \overline{x}^i} = \frac{\partial \mathbf{r}}{\partial x^j} \frac{\partial x^j}{\partial \overline{x}^i} = \frac{\partial x^j}{\partial \overline{x}^i} \mathbf{g}_j.$$

We call them covariant base vectors although the base vectors are coordinate dependent. The base vector $\mathbf{g}^j$ is normal to the coordinate surface $\mathbf{r}(x^1, \ldots, a^j, \ldots, x^n)$. When n=3, the coordinate surfaces are given by $\mathbf{r}(a^1, x^2, x^3)$, $\mathbf{r}(x^1, a^2, x^3)$ and $\mathbf{r}(x^1, x^2, a^3)$, with $a^1, a^2, a^3$ constants.

In Cartesian coordinates $y^i$, a vector $\mathbf{r}$ can be expressed through the Cartesian base $\{\mathbf{e}_i\}$ as $\mathbf{r} = y^i \mathbf{e}_i$. The covariant base vectors are related to the Cartesian base as follows

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial x^i} = \frac{\partial \mathbf{r}}{\partial y^j} \frac{\partial y^j}{\partial x^i} = \frac{\partial y^j}{\partial x^i} \mathbf{e}_j \quad \Rightarrow \quad \mathbf{e}_i = \frac{\partial x^j}{\partial y^i} \mathbf{g}_i. \tag{3.3}$$

The base vectors $\mathbf{g}^i$ are called contravariant base vectors. They are related to the Cartesian base vectors as

$$\mathbf{g}^i = \frac{\partial x^i}{\partial y^j} \mathbf{e}_j \quad \Rightarrow \quad \mathbf{e}_i = \frac{\partial y^i}{\partial x^j} \mathbf{g}^j. \tag{3.4}$$

They obey contravariant like transformation laws

$$\overline{\mathbf{g}}^j = \frac{\partial \overline{x}^j}{\partial y^k} \mathbf{e}_k = \frac{\partial \overline{x}^j}{\partial x^l} \frac{\partial x^l}{\partial y^k} \mathbf{e}_k = \frac{\partial \overline{x}^j}{\partial x^l} \mathbf{g}^l. \tag{3.5}$$

or equivalent

$$\mathbf{g}^i = \frac{\partial x^i}{\partial \overline{x}^j} \overline{\mathbf{g}}^j. \tag{3.6}$$

In Cartesian coordinates, the base vectors are constant everywhere. Generally, a base vector changes when going from a point ${x^i}_0$ to another ${x^i}_0 + dx^j$, for example $d\mathbf{g}_i = (\partial \mathbf{g}_i / \partial x^j) dx^j$. Using (3.3), we have

$$\frac{\partial \mathbf{g}_i}{\partial x^j} = \frac{\partial}{\partial x^j} \left( \frac{\partial y^l}{\partial x^i} \right) \mathbf{e}_l = \frac{\partial^2 y^l}{\partial x^i \partial x^j} \frac{\partial x^m}{\partial y^l} \mathbf{g}_m = \{{}_i{}^m{}_j\} \mathbf{g}_m, \tag{3.7}$$

where $\{{}_i{}^j{}_k\}$, are called the Christoffel symbols of second kind. They are defined as

$$\{{}_i{}^m{}_j\} = \frac{\partial^2 y^l}{\partial x^i \partial x^j} \frac{\partial x^m}{\partial y^l}. \tag{3.8}$$

On the other hand using (3.4), we get

$$\frac{\partial \mathbf{g}_i}{\partial x^j} = \frac{\partial}{\partial x^j}\left(\frac{\partial y^l}{\partial x^i}\right)\mathbf{e}_l = \frac{\partial^2 y^l}{\partial x^i \partial x^j}\frac{\partial y^l}{\partial x^m}\mathbf{g}^m = [ij,m]\mathbf{g}^m \tag{3.9}$$

where $[ij,k]$, the Christoffel symbols of first kind are defined as

$$[ij,m] = \frac{\partial^2 y^l}{\partial x^i \partial x^j}\frac{\partial y^l}{\partial x^m}. \tag{3.10}$$

Notice that both $[ij,m]$ and $\{{_i}{^m}{_j}\}$ are symmetric in ($i$ and $j$). The partial derivatives of $\mathbf{g}_i$ along direction $x^j$ are expressed as linear combinations of the base vectors $\mathbf{g}^l$ and $\mathbf{g}_l$ using the Christoffel symbol of first and second kind.

Taking the derivative of $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$ and using (3.7) we have

$$\frac{\partial(\mathbf{g}_i \cdot \mathbf{g}^j)}{\partial x^k} = 0 \quad \Rightarrow \quad \mathbf{g}_i \cdot \frac{\partial \mathbf{g}^j}{\partial x^k} = -\{{_i}{^j}{_k}\} \quad \Rightarrow \quad \frac{\partial \mathbf{g}^j}{\partial x^k} = -\{{_i}{^j}{_k}\}\mathbf{g}^i.$$

# 4 The metric tensor

The arc length $ds$ between two points in space is an invariant. It must be the same in Cartesian coordiantes $y^i$ and in generalised coordiantes $x^i$. Using (3.3) since

$$ds^2 = dy^i dy^i = \frac{\partial y^i}{\partial x^j}\frac{\partial y^i}{\partial x^k}dx^j dx^k = \mathbf{g}_j \cdot \mathbf{g}_k dx^j dx^k = g_{jk}dx^j dx^k, \tag{4.1}$$

where $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ is in fact a tensor since

$$\overline{g}_{jk} = \frac{\partial y^i}{\partial \overline{x}^j}\frac{\partial y^i}{\partial \overline{x}^k} = \frac{\partial x^l}{\partial \overline{x}^j}\frac{\partial x^m}{\partial \overline{x}^k}\frac{\partial y^i}{\partial x^l}\frac{\partial y^i}{\partial x^m} = \frac{\partial x^l}{\partial \overline{x}^j}\frac{\partial x^m}{\partial \overline{x}^k}g_{lm}.$$

$g_{ij}$ is called the metric tensor. We have

$$g_{ij}\mathbf{g}^j = (\mathbf{g}_i \cdot \mathbf{g}_j)\mathbf{g}^j = \left(\frac{\partial y^p}{\partial x^i}\frac{\partial y^q}{\partial x^j}\mathbf{e}_p \cdot \mathbf{e}_q\right)\frac{\partial x^j}{\partial y^k}\mathbf{e}_k = \frac{\partial y^k}{\partial x^i}\mathbf{e}_k = \mathbf{g}_i.$$

It is expected that the arc length $ds^2 > 0$. To fulfill this, the metric tensor must be positive definite. The Riemann space is a differentiable manifold equipped with a positive definite metric. The length of a vector $A^i$ is $g_{ij}A^i A^j$. For a vector in a space with a positive definite metric, $g_{ij}A^i A^j \geq 0$. For pseudo Riemann space, there is not a requirement that the metric is positive definite and situations can occur where the vector lengths is zero in spite that its components are non-zero. For details see the treatment in [12] chapter 7.

The metric tensor can be used to relate the co and contravariant base vectors through $\mathbf{g}_i = g_{ij}\mathbf{g}^j$. The metric tensor $g_{ij}$ has a reciprocal tensor $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$. They are related through $g_{ij}g^{jk} = \delta_i^k$. In a similar way $\mathbf{g}^i$ can be expressed by $\mathbf{g}_i$ through the relation $\mathbf{g}^i = g^{ij}\mathbf{g}_j$.

From (3.7) and (3.9) we have for the Christoffel symbols

$$\{_i{}^m{}_j\}\mathbf{g}_m = [ij,m]\mathbf{g}^m \quad \Rightarrow \quad [ij,s] = g_{ms}\{_i{}^m{}_j\} \quad \Leftrightarrow \quad \{_i{}^s{}_j\} = g^{ms}[ij,m].$$

A summary of useful relations between the metric tensor and the Christoffel symbols are given below

$$[ij,m] = g_{sm}\{_i{}^s{}_j\}, \tag{4.2}$$

$$\{_i{}^m{}_j\} = g^{sm}[ij,s], \tag{4.3}$$

$$\frac{\partial g_{ij}}{\partial x^k} = [ik,j] + [jk,i], \tag{4.4}$$

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right), \tag{4.5}$$

$$\frac{\partial \mathbf{g}^j}{\partial x^k} = -\{_i{}^j{}_k\}\mathbf{g}^i, \tag{4.6}$$

$$\frac{\partial \mathbf{g}_i}{\partial x^j} = [ij,m]\mathbf{g}^m = \{_i{}^m{}_j\}\mathbf{g}_m. \tag{4.7}$$

In [12] chapter 3, the Christoffel symbols are defined by relation (4.5) and (4.3), which requires the existence of a tensor $g_{ij}$. Note that the Christoffel symbols are not tensors. Using (4.5) and the fact that $g_{ij}$ is a tensor, it can be shown that the Christoffel symbols of first and second kind transform as

$$\overline{[ij,k]} = \frac{\partial^2 x^\gamma}{\partial \overline{x}^i \partial \overline{x}^j}\frac{\partial x^\delta}{\partial \overline{x}^k} g_{\gamma\delta} + \frac{\partial x^\gamma}{\partial \overline{x}^i}\frac{\partial x^\delta}{\partial \overline{x}^j}\frac{\partial x^l}{\partial \overline{x}^k}[\gamma\delta,l], \tag{4.8}$$

$$\overline{\{_i{}^j{}_k\}} = \frac{\partial^2 x^\beta}{\partial \overline{x}^i \partial \overline{x}^k}\frac{\partial \overline{x}^j}{\partial x^\beta} + \frac{\partial \overline{x}^j}{\partial x^\beta}\frac{\partial x^\gamma}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^k}\{_\gamma{}^\beta{}_l\}. \tag{4.9}$$

## 5 Vectors and tensors

A vector $\mathbf{A}$ can be expressed in the base $\{\mathbf{g}_i\}$ or equivalently in the base $\{\mathbf{g}^i\}$ as

$$\mathbf{A} = A_i\mathbf{g}^i = A^j\mathbf{g}_j. \tag{5.1}$$

Given the vector $\mathbf{A}$, the components $A_i$ or $A^i$ can be calculated by the inner product

$$A_i = \mathbf{A}\cdot\mathbf{g}_i \quad \text{and} \quad A^i = \mathbf{A}\cdot\mathbf{g}^i.$$

A physical vector-components are calculated by taking the projection along the normalized base vector $(\mathbf{g}_\alpha/|\mathbf{g}_\alpha|)$. It is easy to show that the physical component of $\mathbf{A}$ is $A(\alpha) = A^\alpha\sqrt{g_{\alpha\alpha}}$, with no sum over $\alpha$.

The co- and contravariant vector components $A_i$ and $A^i$ transform as

$$\overline{A}_i = \frac{\partial x^j}{\partial \overline{x}^i}A_j \tag{5.2}$$

and

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^j}A^j. \tag{5.3}$$

Using (3.6) we get

$$\overline{A}_i\overline{\mathbf{g}}^i = A_j\mathbf{g}^j \quad \Rightarrow \quad \left(\overline{A}_k - A_j\frac{\partial x^j}{\partial \overline{x}^k}\right)\overline{\mathbf{g}}^k = 0 \quad \Rightarrow \quad \overline{A}_k = \frac{\partial x^j}{\partial \overline{x}^k}A_j$$

and similarly for $A^i$ to obtain (5.3).

Assuming the contravariant vector components are given, then the covariant components can be calculated using the following formula and vise versa.

$$A_i = g_{i\sigma}A^\sigma \quad \text{and} \quad A^i = g^{i\sigma}A_\sigma. \tag{5.4}$$

Tensors are generalizations of vectors which are first rank tensors. They can be expressed as a linear combination of dyads. A 2'nd rank tensor $\mathbf{T}$ can be expressed $\mathbf{T} = T_{ij}\mathbf{g}^i\mathbf{g}^j$ where $\{\mathbf{g}^i\mathbf{g}^j\}$ is a dyad base. In 3D space the dyad base has $9$ components. The sets $\{\mathbf{g}_i\mathbf{g}_j\}$, $\{\mathbf{g}_i\mathbf{g}^j\}$, $\{\mathbf{g}^i\mathbf{g}_j\}$ and $\{\mathbf{g}^i\mathbf{g}^j\}$ form bases for the second rank tensor $\mathbf{T}$. The second rank dyadic base in Cartesian coordinates is $\{\mathbf{e}_i\mathbf{e}_j\}$. For example in Cartesian coordinates, the dyad base can be expressed

$$\mathbf{e}_1\mathbf{e}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{e}_2\mathbf{e}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \cdots, \quad \mathbf{e}_3\mathbf{e}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The covariant tensor components can be expressed as

$$T_{ij} = \mathbf{g}_i \cdot \mathbf{T}\mathbf{g}_j$$

and for example the mixed components $T_i^{\ j}$ are

$$T_i^{\ j} = \mathbf{g}_i \cdot \mathbf{T}\mathbf{g}^j.$$

Tensor components can be expressed in various combinations of bases, for example $T_{ij}$ and $T^{ij}$ called covariant and contravariant tensors, $T^i_{\ j}$ and $T_i^{\ j}$ are called mixed tensors.

Notice that the components $T_i^{\ j}$ and $T^j_{\ i}$ are equal only if the tensor is symmetric. As for vectors, conversions from covariant to contravariant components can be done using the expressions analog to (5.4)

$$T_i^{\ j} = g_{il}T^{lj} \quad \text{and} \quad T_{ij} = g_{ik}g_{jl}T^{kl} \tag{5.5}$$

and so on...

# 6 Derivation of vectors and tensors

Consider a curve $\mathcal{C}(t)$ in the space $X_n$. Let $t$ be a parameter (e.g. the arc length). We want to differentiate scalar, vector or tensor fields along $\mathcal{C}$ which is expressed by $\mathbf{r} = \mathbf{r}(t)$ which in component form is $x^i = x^i(t)$. The field $\mathbf{t} = d\mathbf{r}/dt$ is the tangent of the curve $\mathcal{C}$, which on component form is given as

$$t^i = \frac{dx^i}{dt}. \tag{6.1}$$

Let $\alpha$ be a scalar field. The derivative of $\alpha$ along $\mathcal{C}$ is simply

$$\frac{d\alpha}{dt} = \frac{\partial\alpha}{\partial x^i}\frac{dx^i}{dt} = \frac{\partial\alpha}{\partial x^i}t^i, \tag{6.2}$$

which also can be written on the form

$$\frac{d\alpha}{dt} = \nabla\alpha \cdot \mathbf{t}.$$

To calculate the derivative of a vector $\mathbf{v}$ along the curve $\mathcal{C}$ is a more complicated process than ordinary derivation. It is not sufficient to consider only the change of the components of $\mathbf{v}$ along $\mathcal{C}$, but one has also to take into account the change of the base vectors along $\mathcal{C}$. This can be expressed as follows

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial x^k}\frac{dx^k}{dt} = \left(\frac{\partial v_i}{\partial x^k}\mathbf{g}^i + v_i\frac{\partial\mathbf{g}^i}{\partial x^k}\right)\frac{dx^k}{dt} = \left(\frac{\partial v^i}{\partial x^k}\mathbf{g}_i + v^i\frac{\partial\mathbf{g}_i}{\partial x^k}\right)\frac{dx^k}{dt}. \tag{6.3}$$

From (6.3), (3.7) and (3.9), the derivative of $\mathbf{v}$ along the curve $\mathcal{C}$ become

$$\frac{d\mathbf{v}}{dt} = \left(\frac{\partial v_i}{\partial x^k} - v_l\{{}_i{}^l{}_k\}\right)\frac{dx^k}{dt}\mathbf{g}^i = \left(\frac{\partial v^i}{\partial x^k} + v^l\{{}_l{}^i{}_k\}\right)\frac{dx^k}{dt}\mathbf{g}_i. \tag{6.4}$$

The quantities

$$\frac{\partial v_i}{\partial x^k} - v_l\{{}_i{}^l{}_k\}$$

and

$$\frac{\partial v^i}{\partial x^k} + v^l\{{}_l{}^i{}_k\},$$

are in fact tensors. They are defined as the partial covariant derivatives, simply denoted covariant derivatives of $v_i$ and $v^i$ respectively and are written $v_{i|k}$ and $v^i{}_{|k}$. We have

$$v_{i|k} = \frac{\partial v_i}{\partial x^k} - v_l\{{}_i{}^l{}_k\} \tag{6.5}$$

and

$$v^i{}_{|k} = \frac{\partial v^i}{\partial x^k} + v^l\{{}_l{}^i{}_k\}. \tag{6.6}$$

The derivative along the curve expressed by the covariant derivatives are

$$\frac{d\mathbf{v}}{dt} = v_{i|k}\mathbf{g}^i\frac{dx^k}{dt} = v^i{}_{|k}\mathbf{g}_i\frac{dx^k}{dt}.$$

The derivative of $\mathbf{v}$ along the curve $\mathcal{C}$ can be expressed by the absolute derivative $\delta v^i/\delta t$ as

$$\frac{d\mathbf{v}}{dt} = \frac{\delta v^i}{\delta t}\mathbf{g}_i. \tag{6.7}$$

From (6.7) and (6.4) it follows that

$$\frac{\delta v^i}{\delta t} = \frac{dv^i}{dt} + v^l \{{_l}{^i}{_j}\} t^j. \tag{6.8}$$

The covariant derivatives of vectors (and tensors) are constructed so they transform as tensors. Using (6.5) and (4.9) we get

$$\overline{v_{i|k}} = \frac{\partial \overline{v}_i}{\partial \overline{x}^k} - \overline{v}_l \overline{\{{_i}{^l}{_k}\}} = \frac{\partial x^j}{\partial \overline{x}^i} \frac{\partial x^l}{\partial \overline{x}^k} \left( \frac{\partial v_j}{\partial x^l} - v_s \{{_j}{^s}{_l}\} \right) = \frac{\partial x^j}{\partial \overline{x}^i} \frac{\partial x^l}{\partial \overline{x}^k} v_{j|l}.$$

It can be shown that the derivative of a covariant second rank tensor $c_{ij}$ is

$$c_{ij|k} = \frac{\partial c_{ij}}{\partial x^k} - c_{lj} \{{_i}{^l}{_k}\} - c_{il} \{{_j}{^l}{_k}\} \tag{6.9}$$

and the derivative of a mixed tensor is

$$c^i{}_{j|k} = \frac{\partial c^i{}_j}{\partial x^k} + c^l{}_j \{{_l}{^i}{_k}\} - c^i{}_l \{{_j}{^l}{_k}\}. \tag{6.10}$$

By applying (6.9), the covariant derivative of a product of two vectors can be shown to follow the product rule for for ordinary derivation

$$\begin{aligned} (a_i b_j)_{|k} = \;& \tfrac{\partial a_i b_j}{\partial x^k} - a_l b_j \{{_i}{^l}{_k}\} - a_i b_l \{{_j}{^l}{_k}\} = \\ & a_i \left( \tfrac{\partial b_j}{\partial x^k} - b_l \{{_j}{^l}{_k}\} \right) + \left( \tfrac{\partial a_i}{\partial x^k} - a_l \{{_i}{^l}{_k}\} \right) b_j = \\ & \qquad a_{i|k} b_j + a_i b_{j|k}. \end{aligned} \tag{6.11}$$

The covariant derivative of a scalar field equals its partial derivative, $\alpha_{|i} = \partial \alpha / x^i$. This is due to the fact that a scalar field has no directional information. Write $\alpha = \mathbf{a} \cdot \mathbf{b} = a_i b^i$ and take the covariant derivative

$$\begin{aligned} \alpha_{|i} = \;& (a_k b^k)_{|i} = a_{k|i} b^k + a_k b^k{}_{|i} = \\ & \tfrac{\partial a_k}{\partial x^i} b^k + a_k \tfrac{\partial b^k}{\partial x^i} - a_l b^k \{{_k}{^l}{_i}\} + a_k b^l \{{_l}{^k}{_i}\} = \\ & \qquad \tfrac{\partial (a_k b^k)}{\partial x^i} = \tfrac{\partial \alpha}{\partial x^i}. \end{aligned}$$

The covariant derivative of the metric tensor is zero. From (6.9), (4.2) and (4.4) we have

$$g_{ij|k} = \frac{\partial g_{ij}}{\partial x^k} - g_{il} \{{_j}{^l}{_k}\} - g_{lj} \{{_i}{^l}{_k}\} = \frac{\partial g_{ij}}{\partial x^k} - [jk, i] - [ik, j] = 0,$$

whis is known as Ricci's lemma. We may also expect that $g^{ij}{}_{|k} = 0$. Let us start showing that $\delta^i_{j|k} = 0$. First showing that $\delta^i_j$ is a tensor. From (2.3) and (2.4)

$$\overline{\delta}^k_l = \frac{\partial \overline{x}^k}{\partial x^j} \frac{\partial x^j}{\partial \overline{x}^l} = \frac{\partial \overline{x}^k}{\partial x^i} \frac{\partial x^i}{\partial x^j} \frac{\partial x^j}{\partial \overline{x}^l} = \frac{\partial \overline{x}^k}{\partial x^i} \frac{\partial x^i}{\partial \overline{x}^s} \frac{\partial \overline{x}^s}{\partial x^j} \frac{\partial x^j}{\partial \overline{x}^l} = \frac{\partial \overline{x}^k}{\partial x^i} \frac{\partial x^j}{\partial \overline{x}^l} \delta^i_j,$$

then using the rule for differentiation of a mixed tensor (6.10)

$$\delta^i{}_{j|k} = \frac{\partial \delta^i{}_j}{\partial x^k} + \delta^\sigma{}_j \{{_\sigma}{^i}{_k}\} - \delta^i{}_\sigma \{{_j}{^\sigma}{_k}\} = 0.$$

Taking the derivative of $g_{ij}g^{jk} = \delta^k{}_i$ gives $g^{ij}{}_{|k} = 0$.

Changing from co- to contravariant components can be done by raising and lowering the indices. Using Ricci's lemma and the product rule for covariant differentiation

$$a^i = g^{si}a_s \quad \Rightarrow \quad a^i{}_{|j} = (g^{si}a_s)_{|j} = g^{si}a_{s|j}$$

and similarly

$$a^{i|j} = g^{si}g^{\sigma j}a_{s|\sigma}.$$

All components $a_{i|j}, a^i{}_{|j}, a_{i|}{}^j$ and $a^{i|j}$ are equivalent. Notice that although we write $a^{i|j}$ the term contravariant differentiation is not used.

# 7 Cylinder coordinates, basic expressions

Most expressions involving differentiation of vectors, like $\text{grad}(\alpha), \text{div}(\mathbf{a}), \text{curl}(\mathbf{a})$ etc..., can be found in books of mathematical formulas ([14]). Expressions involving for example tensor components and terms derived from them can not be found in standard collections of formulas. An example of such a term is the double divergence term of the momentum flux density tensor appearing in Lighthill's equation $\nabla \cdot (\nabla \cdot \rho\mathbf{v}\mathbf{v})$. An attempt to find this term on the web was not successful and we had to calculate it from the basics. There are many examples of such terms which implies that we have to compute them the hard way as explored below.

Let curvilinear coordinates in $\mathbf{R}^3$ be denoted by $(x^1, x^2, x^3)$. In the case of cylinder coordinates we write $(R, \theta, z)$, the mapping between cylinder coordinates and Cartesian coordinates $(y^1, y^2, y^3)$ is

| Cylinder coordinates | Cartesian coordinates |
|---|---|
| $x^1 = R$ | $y^1 = R\cos\theta$ |
| $x^2 = \theta$ | $y^2 = R\sin\theta$ |
| $x^3 = z$ | $y^3 = z$ |

Let the unit base vectors in Cartesian coordinates be $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The unit base vectors in cylinder coordinates can be written as $\{\mathbf{e}_R, \mathbf{e}_\theta, \mathbf{e}_z\}$. A vector can be expressed by its physical components as follows

$$\mathbf{a} = a_R\mathbf{e}_R + a_\theta\mathbf{e}_\theta + a_z\mathbf{e}_z.$$

The unit base vectors for cylinder coordinates can be expressed in Cartesian coordinates as

$$\begin{aligned}
&\mathbf{e}_R(\theta) = \mathbf{e}_1\cos(\theta) + \mathbf{e}_2\sin(\theta),\\
&\mathbf{e}_\theta(\theta) = -\mathbf{e}_1\sin(\theta) + \mathbf{e}_2\cos(\theta),\\
&\mathbf{e}_z = \mathbf{e}_z,
\end{aligned}$$

where

$$\partial\mathbf{e}_R/\partial\theta = \mathbf{e}_\theta \quad \text{and} \quad \partial\mathbf{e}_\theta/\partial\theta = -\mathbf{e}_R. \tag{7.1}$$

Let $\mathbf{r}(x^1, c_2, c_3)$ where $c_2$ and $c_3$ are constants express the coordinate line $x^1$, $\mathbf{r}(c_1, x^2, c_3)$ the coordinate line $x^2$ etc..., then the tangent base vector along the line $x^i$ is $\mathbf{g}_i = \partial\mathbf{r}/\partial x^i$. The reciprocal or normal base vectors $\mathbf{g}^i$ satisfy $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$. Any vector $\mathbf{r}$ can be expressed in cylinder coordinates as

$$\mathbf{r} = R\mathbf{e}_R(\theta) + z\mathbf{e}_z. \tag{7.2}$$

The base vectors $\mathbf{g}_i$ and $\mathbf{g}^i$ then become

$$\begin{aligned} \mathbf{g}_1 &= \mathbf{e}_R, & \mathbf{g}^1 &= \mathbf{e}_R, \\ \mathbf{g}_2 &= R\mathbf{e}_\theta, & \mathbf{g}^2 &= (1/R)\mathbf{e}_\theta, \\ \mathbf{g}_3 &= \mathbf{e}_z, & \mathbf{g}^3 &= \mathbf{e}_z. \end{aligned} \tag{7.3}$$

The components of the metric tensors $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ and the reciprocal $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$ become

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/R^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.4}$$

The Christoffel symbol of first kind $[ij,k]$ and of second kind $\{{}_i{}^j{}_k\}$, can be calculated from (7.4) using (4.5) and (4.3). All components become zero except

$$[12,2] = [21,2] = R, \quad [22,1] = -R, \tag{7.5}$$

$$\{{}_1{}^2{}_2\} = \{{}_2{}^2{}_1\} = 1/R, \quad \{{}_2{}^1{}_2\} = -R. \tag{7.6}$$

Consider a vector field $\mathbf{a}$. It can equivalently be expressed expressed by the physical, co and contravariant components as

$$\begin{aligned} \mathbf{a} &= a_R\mathbf{e}_R + a_\theta\mathbf{e}_\theta + a_z\mathbf{e}_z \\ &= a_1\mathbf{e}_R + a_2\mathbf{e}_\theta/R + a_3\mathbf{e}_z \\ &= a^1\mathbf{e}_R + a^2R\mathbf{e}_\theta + a^3\mathbf{e}_z. \end{aligned}$$

Since $\{\mathbf{e}_R, \mathbf{e}_\theta, \mathbf{e}_z\}$ are unit vectors, we get the following relations between the physical components and the covariant and contravariant components

$$\begin{aligned} a_R &= a(1) = a_1 = a^1, \\ a_\theta &= a(2) = a_2/R = Ra^2, \\ a_z &= a(3) = a_3 = a^3. \end{aligned} \tag{7.7}$$

The physical components can in the case of orthogonal coordinates be calculated from the expression $a(\alpha) = \sqrt{g_{\alpha\alpha}}a^\alpha$, where Greek letters in this case means no-sum. The physical components of a second rank tensor is $A(\alpha\beta) = \sqrt{g_{\alpha\alpha}g_{\beta\beta}}A^{\alpha\beta}$. Using this expression, the contravariant components of a second rank tensor in cylinder coordinates can be expressed through the physical components and vice versa

$$(A^{ij}) = \begin{pmatrix} A_{rr} & A_{r\theta}/R & A_{rz} \\ A_{\theta r}/R & A_{\theta\theta}/R^2 & A_{\theta r}/R \\ A_{zr} & A_{z\theta}/R & A_{zz} \end{pmatrix}, \tag{7.8}$$

and the relation between the co and contravariant components are calculated by (5.5) to yield

$$(A_{ij}) = \begin{pmatrix} A^{11} & R^2A^{12} & A^{13} \\ R^2A^{21} & R^4A^{22} & R^2A^{23} \\ A^{31} & R^2A^{32} & A^{33} \end{pmatrix} = \begin{pmatrix} A_{rr} & RA_{r\theta} & A_{rz} \\ RA_{\theta r} & R^2A_{\theta\theta} & RA_{\theta r} \\ A_{zr} & RA_{z\theta} & A_{zz} \end{pmatrix}. \tag{7.9}$$

## 8 Covariant derivatives in cylinder coordinates

By using the definition (6.5) and (6.6) together with (7.6), we can calculate the components of the covariant derivative tensors. Expressed by the physical components $(a_R, a_\theta, a_z)$, we obtain

$$(a_{i|j}) = \begin{pmatrix} \frac{\partial a_R}{\partial R} & \frac{\partial a_R}{\partial \theta} - a_\theta & \frac{\partial a_R}{\partial z} \\ R\frac{\partial a_\theta}{\partial R} & R\left(\frac{\partial a_\theta}{\partial \theta} + a_R\right) & R\frac{\partial a_\theta}{\partial z} \\ \frac{\partial a_z}{\partial R} & \frac{\partial a_z}{\partial \theta} & \frac{\partial a_z}{\partial z} \end{pmatrix} \tag{8.1}$$

The mixed components $a^i{}_{|j}$ of the tensor can be calculated by using $a^i{}_j = g^{si}a_{sj}$, we obtain

$$(a^i{}_{|j}) = \begin{pmatrix} \frac{\partial a_R}{\partial R} & \frac{\partial a_R}{\partial \theta} - a_\theta & \frac{\partial a_R}{\partial z} \\ \frac{1}{R}\frac{\partial a_\theta}{\partial R} & \frac{1}{R}\left(\frac{\partial a_\theta}{\partial \theta} + a_R\right) & \frac{1}{R}\frac{\partial a_\theta}{\partial z} \\ \frac{\partial a_z}{\partial R} & \frac{\partial a_z}{\partial \theta} & \frac{\partial a_z}{\partial z} \end{pmatrix} \tag{8.2}$$

The component sets $a_{i|}{}^j$ and $a^{i|j}$ be calculated in the same way so we do not write them up here.

## 9 Vector operations

From now, we use the convention that the partial derivative of a scalar is written $\partial\alpha/\partial x^i = \alpha_{,i}$ and the partial derivative of a vector $\partial\mathbf{a}/\partial x^i = \mathbf{a}_{,i}$. As an example it is convenient to express the divergence of a vector $\nabla\cdot\mathbf{a} = \mathbf{g}^i\cdot\mathbf{a}_{,i} = \mathbf{g}^i\cdot(\mathbf{g}_k a^k)_{,i} = a^i{}_{|i}$. Some of the most common scalar and vector operations can then be expressed

$$\nabla\alpha \equiv \mathbf{g}^i\alpha_{,i}\,, \tag{9.1}$$

$$\nabla\cdot\mathbf{a} \equiv \mathbf{g}^i\cdot\mathbf{a}_{,i} = a^i{}_{|i} = \frac{1}{\sqrt{g}}(\sqrt{g}\,a^i)_{,i}\,, \tag{9.2}$$

$$\nabla\times\mathbf{a} \equiv \mathbf{g}^i\times\mathbf{a}_{,i} = \varepsilon^{ijk}a_{k,j}\mathbf{g}_i = \frac{1}{\sqrt{g}}\epsilon^{ijk}a_{k,j}\mathbf{g}_i = \omega^i\mathbf{g}_i, \tag{9.3}$$

$$\nabla^2\alpha = \nabla\cdot\nabla\alpha = \alpha_{|i|}{}^i = \frac{1}{\sqrt{g}}(\sqrt{g}\,g^{ij}\alpha_{,j}\,)_{,i}\,, \tag{9.4}$$

$$\nabla^2\mathbf{a} = \nabla\cdot\nabla\mathbf{a} = a^{i|k}_k\mathbf{g}_i = (\frac{1}{\sqrt{g}}(\sqrt{g}\,a^{i|k})_{,k} + a^{l|k}\{{}_l{}^i{}_k\})\mathbf{g}_i. \tag{9.5}$$

Notice that $\nabla\cdot\mathbf{a}$ is a scalar while $\nabla\mathbf{a} = \mathbf{g}^i a_{i|j}\mathbf{g}^j$ is a second rank tensor.

Examples of various vector derivative operations.

$\nabla\alpha$:
The gradient of the scalar $\alpha$ is $\nabla\alpha = \mathbf{g}^i\alpha,_i = \mathbf{g}^i\alpha_{|i}$.
In cylinder coordinates

$$\nabla\alpha = \alpha,_R\,\mathbf{g}^1 + \alpha,_\theta\,\mathbf{g}^2 + \alpha,_z\,\mathbf{g}^3 = \frac{\partial\alpha}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial\alpha}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\alpha}{\partial z}\mathbf{e}_z.$$

$\nabla\cdot\mathbf{a}$:
The divergence of a vector is the contraction of the covariant derivative tensor formed by that vector: $\nabla\cdot\mathbf{a} = \mathrm{tr}(\nabla\mathbf{a}) = a^i{}_{|i}$.
We use that $g^{ij} = Ko(g_{ij})/g$ and $\partial g/\partial g_{ij} = Ko(g_{ij})$, so $g,_i = gg^{kl}g_{kl},_i$. We have $g_{kl},_i = [ki,l] + [li,k]$. Then $g,_i = gg^{kl}([ki,l]+[li,k]) = 2g\{_k{}^k{}_i\}, \Rightarrow \{_k{}^k{}_i\} = g,_i/2g = (1/\sqrt{g})(\sqrt{g}),_i$.
Then $a^i{}_{|i} = a^i,_i + a^s\{_s{}^i{}_i\} = (1/\sqrt{g})(\sqrt{g}a^i),_i$.
In cylinder coordinates, using the above formula

$$\nabla\cdot\mathbf{a} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial a^1}{\partial R}\right) + \frac{\partial a^2}{\partial\theta} + \frac{\partial a^3}{\partial z} = \frac{1}{R}\frac{\partial(Ra_R)}{\partial R} + \frac{1}{R}\frac{\partial a_\theta}{\partial\theta} + \frac{\partial a_z}{\partial z}.$$

$\nabla\times\mathbf{a}$:
The curl becomes: $\nabla\times\mathbf{a} = \omega^i\mathbf{g}_i = \varepsilon^{ijk}a_{k|j}\mathbf{g}_i = \varepsilon^{ijk}a_k,_j\,\mathbf{g}_i = (1/\sqrt{g})\epsilon^{ijk}a_{k,j}\mathbf{g}_i$, since $\{_k{}^s{}_j\}$ is symmetric in $kj$. The contravariant components of the curl become

$$\begin{aligned}
\omega^1 &= \frac{1}{R}\frac{\partial a_z}{\partial\theta} - \frac{\partial a_\theta}{\partial z} = \omega_R,\\
\omega^2 &= \frac{1}{R}\left(\frac{\partial a_R}{\partial z} - \frac{\partial a_z}{\partial R}\right) = \frac{\omega_\theta}{R},\\
\omega^3 &= \frac{1}{R}\frac{\partial(Ra_\theta)}{\partial R} - \frac{1}{R}\frac{\partial a_R}{\partial\theta} = \omega_z,
\end{aligned}$$

where $\{\omega_R, \omega_\theta, \omega_z\}$ are the physical components of $\nabla\times\omega$.

$\nabla^2\alpha$:
The Laplaceian of an absolute scalar is $\nabla^2\alpha = \nabla\cdot\nabla\alpha = \nabla\cdot(\mathbf{g}^i\alpha_{|i}) = \nabla\cdot(\mathbf{g}^i\alpha_{,i})$. Let $a_i = \alpha_{|i}$, then $\nabla\cdot\nabla\alpha = \nabla\cdot\mathbf{a} = a^i{}_{|i} = (g^{ik}\alpha_{|k})_{|i} = g^{ik}{}_{|i}\alpha_{|k} + g^{ik}\alpha_{|ik} == \alpha_{|i|}{}^i = \alpha|^i_i$. Set $a^i = \alpha^{|i}$, then $\nabla^2\alpha = \nabla\cdot(\nabla\alpha) = \nabla\cdot\mathbf{a} = (1/\sqrt{g})(\sqrt{g}a^i),_i$. Now $a^i = (\nabla\alpha)^i = g^{ij}\alpha,_j$, which implies $\nabla^2\alpha = (1/\sqrt{g})(\sqrt{g}g^{ij}\alpha,_j),_i$. Then in cylinder coordinates $\nabla^2\alpha$ becomes

$$\begin{aligned}
\nabla^2\alpha &= \frac{1}{R}\left((Rg^{ij}\alpha,_j),_R + (Rg^{2j}\alpha,_j),_\theta + (Rg^{3j}\alpha,_j),_z\right)\\
&= \frac{1}{R}\left(R\alpha,_R\right),_R + \frac{\alpha,_{\theta\theta}}{R^2} + \alpha,_{zz}.
\end{aligned}$$

$\nabla^2\mathbf{a}$:
The Laplaceian of a vector is: $\nabla^2\mathbf{a} = \nabla\cdot\nabla\mathbf{a}$. Note that in general $\nabla(\nabla\cdot\mathbf{a}) \neq \nabla\cdot(\nabla\mathbf{a})$. We take $\nabla^2\mathbf{a} = \nabla\cdot(\nabla\mathbf{a})$. Where $\nabla\mathbf{a}$ is a second rank tensor. Let us write $\nabla\mathbf{a} = \mathbf{T}$, then $\nabla\cdot\nabla\mathbf{a} = \nabla\cdot\mathbf{T} = T^{ik}_{|k}\mathbf{g}_i$. Now selecting $T^{ik} = a^{i|k}$ and using $\nabla\cdot\nabla\mathbf{a} = (a^{i|k})_{|k}\mathbf{g}_i = a^i|^k_k\mathbf{g}_i$.
The covariant derivative of the tensor $a^{i|k}$ becomes

$$(a^{i|k})_{|k} = (a^{i|k}),_k + a^{\sigma|k}\{_\sigma{}^k{}_i\} + a^{i|\sigma}\{_\sigma{}^k{}_k\} = (1/\sqrt{g})(\sqrt{g}a^{i|k}),_k + a^{\sigma|k}\{_\sigma{}^k{}_i\}.$$

Using the expressions for the Christoffel symbols in cylinder coordinates, we get

$$a^i|_k^k = (a^{i|k})_{|k} = (a^{i|k})_{,k} + \{_k{}^l{}_l\}a^{i|k} + \{_l{}^k{}_i\}a^{l|k},$$

which gives

$$\begin{aligned}
a^1|_k^k &= \frac{1}{R}a^1_{|1} + (a^1_{|1})_{,R} + \frac{1}{R^2}(a^1_{|2})_{,\theta} + (a^1_{|3})_{,z} - \frac{1}{R}a^2_{|2} \\
&= \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial a_R}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 a_R}{\partial \theta^2} + \frac{\partial^2 a_R}{\partial z^2} - \frac{a_R}{R^2} - \frac{2}{R^2}\frac{\partial a_\theta}{\partial \theta} \\
&= (\nabla^2 \mathbf{a})_R, \\
a^2|_k^k &= \frac{1}{R}a^2_{|1} + (a^2_{|1})_{,R} + \frac{1}{R^2}(a^2_{|2})_{,\theta} + (a^2_{|3}), z + \frac{1}{R}\left(a^2_{|1} + \frac{1}{R^2}a^1_{|2}\right) \\
&= \frac{1}{R}\left\{\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial a_\theta}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 a_\theta}{\partial \theta^2} + \frac{\partial^2 a_\theta}{\partial z^2} + \frac{2}{R^2}\frac{\partial a_R}{\partial \theta} - \frac{a_\theta}{R^2}\right\} \\
&= \frac{1}{R}(\nabla^2 \mathbf{a})_\theta, \\
a^3|_k^k &= \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial a_z}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 a_z}{\partial \theta^2} + \frac{\partial^2 a_z}{\partial z^2} \\
&= (\nabla^2 \mathbf{a})_z.
\end{aligned}$$

## 10 Tensor operations

Let $\mathbf{u}$ be the velocity vector. Note that $\partial u_i/\partial x^j$ does not transform as a tensor, but $u_{i|j}$ does. We call it the velocity derivative tensor. It can be written

$$u_{i|j} = (u_{i|j} + u_{j|i})/2 + (u_{i|j} - u_{j|i})/2 = S_{ij} + \omega_{ij}/2$$

where we have introduced the rate of strain tensor

$$S_{ij} = (u_{i|j} + u_{j|i})/2, \tag{10.1}$$

and the rotation tensor

$$\omega_{ij} = (u_{i|j} - u_{j|i}) = (u_{i,j} - u_{j,i}). \tag{10.2}$$

The covariant components of the rate of strain tensor in cylinder coordinates are

$$(S_{ij}) = \begin{pmatrix} \frac{\partial u_R}{\partial R} & \left(\frac{\partial u_R}{\partial \theta} - u_\theta + R\frac{\partial u_\theta}{\partial R}\right)/2 & \left(\frac{\partial u_z}{\partial R} + \frac{\partial u_R}{\partial z}\right)/2 \\ - & R\left(\frac{\partial u_\theta}{\partial \theta} + u_R\right) & \left(R\frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{\partial \theta}\right)/2 \\ - & - & \frac{\partial u_z}{\partial z} \end{pmatrix}$$

The physical components of a second rank tensor is $T(\alpha\beta) = \sqrt{g_{\alpha\alpha}g_{\beta\beta}}T^{\alpha\beta}$, where Greek letters in this case means no-sum. The physical components of the rate of strain tensor expressed by the physical components of the velocity, with $i, j = R, \theta, z$ we get

$$S(ij) = \begin{pmatrix} \frac{\partial u_R}{\partial R} & \left(\frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R} + \frac{1}{R}\frac{\partial u_R}{\partial \theta}\right)/2 & \left(\frac{\partial u_z}{\partial R} + \frac{\partial u_R}{\partial z}\right)/2 \\ - & \frac{1}{R}\left(\frac{\partial u_\theta}{\partial \theta} + u_R\right) & \left(\frac{1}{R}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right)/2 \\ - & - & \frac{\partial u_z}{\partial z} \end{pmatrix}. \tag{10.3}$$

Sometimes it can be of practical interest to express the components of a symmetric tensor in Cartesian coordinates $\{y^i\}$, we denote it $s_{ij}$, derived from its components $S_{ij}$ in generalized coordinates $\{x^i\}$. We have

$$s_{ij} = \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j} S_{kl}.$$

Using the covariant components of a symmetric tensor $S_{ij}$ in cylinder coordinates, we get the following Cartesian components

$$\begin{aligned}
s_{11} &= (\cos^2\theta)S_{11} - (\sin(2\theta)/R)S_{12} + (\sin^2\theta/R^2)S_{22},\\
s_{12} &= s_{21} = (\sin(2\theta)/2)S_{11} + (\cos(2\theta)/R)S_{12} - (\sin(2\theta)/2R^2)S_{22},\\
s_{13} &= s_{31} = (\cos\theta)S_{13} - (\sin\theta/R)S_{23},\\
s_{22} &= (\sin^2\theta)S_{11} + (\sin(2\theta)/R)S_{12} + (\cos^2\theta/R^2)S_{22},\\
s_{23} &= s_{32} = (\sin\theta)S_{13} + (\cos\theta/R)S_{23},\\
s_{33} &= S_{33}.
\end{aligned}$$

These expressions may be useful when visualizing symmetric tensors where the components are given in for example cylinder coordinates and the coordinate system used by the visualization system is Cartesian.

# 11 Rotating coordinates

Consider two orthogonal coordinate systems $O$ and $\overline{O}$ with common origin with basis vectors $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ and $\{\overline{\mathbf{g}}_1, \overline{\mathbf{g}}_2, \overline{\mathbf{g}}_3\}$. Let $\mathbf{g}_3 = \overline{\mathbf{g}}_3$ be the rotation axis that coinside for the two systems. Assume $O$ is an inertial sysem while $\overline{O}$ is rotating with angular velocity $\Omega$, where $\boldsymbol{\Omega} = \Omega \mathbf{g}_3$. Then, $\mathbf{g}_i$ are time independent while $\overline{\mathbf{g}}_i = \overline{\mathbf{g}}_i(t)$ are time dependent.

$$\begin{aligned}
\overline{\mathbf{g}}_1 &= \mathbf{g}_1 \cos\Omega t + \mathbf{g}_2 \sin\Omega t\\
\overline{\mathbf{g}}_2 &= -\mathbf{g}_1 \sin\Omega t + \mathbf{g}_2 \cos\Omega t
\end{aligned} \tag{11.1}$$

A point $P$ has position given by the vector $\mathbf{r}$ with coordinates $x^i$ and $\overline{x}^i$ with $i = 1, 2, 3$. Then

$$\mathbf{r} = x^i \mathbf{g}_i = \overline{x}^i \overline{\mathbf{g}}_i$$

which implies

$$\begin{aligned}
x^1 &= \overline{x}^1 \cos\Omega t - \overline{x}^2 \sin\Omega t\\
x^2 &= \overline{x}^1 \sin\Omega t + \overline{x}^2 \cos\Omega t,
\end{aligned} \tag{11.2}$$

or equivalent

$$\begin{aligned}
\overline{x}^1 &= x^1 \cos\Omega t + x^2 \sin\Omega t\\
\overline{x}^2 &= -x^1 \sin\Omega t + x^2 \cos\Omega t,
\end{aligned}$$

which also can be written

$$\begin{aligned}
\overline{x}^1 &= x^1 \cos(-\Omega t) - x^2 \sin(-\Omega t)\\
\overline{x}^2 &= x^1 \sin(-\Omega t) + x^2 \cos(-\Omega t).
\end{aligned} \tag{11.3}$$

The time derivative of (11.1) becomes

$$\dot{\overline{\mathbf{g}}}_1 = \Omega(-\mathbf{g}_1 \sin \Omega t + \mathbf{g}_2 \cos \Omega t) = \Omega \overline{\mathbf{g}}_2$$
$$\dot{\overline{\mathbf{g}}}_2 = \Omega(-\mathbf{g}_1 \cos \Omega t - \mathbf{g}_2 \sin \Omega t) = -\Omega \overline{\mathbf{g}}_1$$

With rotation vector $\mathbf{\Omega} = \Omega \mathbf{g}_3$, the above expression can be written in compact form

$$\dot{\overline{\mathbf{g}}}_i = \mathbf{\Omega} \times \overline{\mathbf{g}}_i \tag{11.4}$$

The time varying vector $\mathbf{a}(t)$ given in the rotating frame is $\mathbf{a}(t) = \overline{a}^i(t)\overline{\mathbf{g}}_i$. Computing the time derivative of $\mathbf{a}(t)$ gives

$$\dot{\mathbf{a}} = \dot{\overline{a}}^i\, \overline{\mathbf{g}}_i + \overline{a}^i \dot{\overline{\mathbf{g}}}_i = \dot{\overline{a}}^i\, \overline{\mathbf{g}}_i + \mathbf{\Omega} \times (\overline{a}^i\, \overline{\mathbf{g}}_i). \tag{11.5}$$

The velocity vector is $\mathbf{v} = \dot{x}^i \mathbf{g}_i$, (11.5) gives then the relation between the velocity components in the inertial system $O$ and the rotating system $\overline{O}$

$$\dot{x}^i \mathbf{g}_i = \dot{\overline{x}}^i\, \overline{\mathbf{g}}_i + \Omega \times (\overline{x}^i\, \overline{\mathbf{g}}_i) \tag{11.6}$$

which states that the velocity in the frame $O$ equals the velocity observed in $\overline{O}$ plus the contribution from the rotation of $\overline{O}$.

( Notice that $x^i \mathbf{g}_i = \overline{x}^i\, \overline{\mathbf{g}}_i$, points to the same point $P$. For any instants $t$, $\mathbf{g}_i$ follow the transformation law $\overline{\mathbf{g}}_i = \partial x^k / \partial \overline{x}^i \mathbf{g}_k$. but that although the velocity $\mathbf{v} = \dot{x}^i \mathbf{g}_i = v^i \mathbf{g}_i$ is a vector, as the frame $\overline{O}$ is accelerated, $v_i \mathbf{g}_i \neq \overline{v}_i \overline{\mathbf{g}}_i$. In fact the velocity $v_i$ that follows $(11.6)$ is not a tensorial neither is the acceleration $a_i$ that we derive below. )

The acceleration is obtained by taking the time derivative of (11.6) and using (11.4). The acceleration in $O$ is expressed by quantities in the rotating system $\overline{O}$ as

$$\ddot{x}^i \mathbf{g}_i = \ddot{\overline{x}}^i\, \overline{\mathbf{g}}_i + \dot{\overline{x}}^i \mathbf{\Omega} \times \overline{\mathbf{g}}_i + \dot{\mathbf{\Omega}} \times (\overline{x}^i\, \overline{\mathbf{g}}_i) + \mathbf{\Omega} \times (\dot{\overline{x}}^i\, \overline{\mathbf{g}}_i) + \mathbf{\Omega} \times (\overline{x}^i (\mathbf{\Omega} \times \overline{\mathbf{g}}_i)), \tag{11.7}$$

which finally becomes

$$\ddot{x}^i \mathbf{g}_i = \ddot{\overline{x}}^i\, \overline{\mathbf{g}}_i + 2\mathbf{\Omega} \times \dot{\overline{x}}^i\, \overline{\mathbf{g}}_i + \dot{\mathbf{\Omega}} \times (\overline{x}^i\, \overline{\mathbf{g}}_i) + \mathbf{\Omega} \times (\mathbf{\Omega} \times \overline{x}^i\, \overline{\mathbf{g}}_i). \tag{11.8}$$

Here the following expressions are valid in the rotating frame $\overline{O}$, where

- $\ddot{\overline{x}}^i\, \overline{\mathbf{g}}_i$ is the acceleration,
- $2\mathbf{\Omega} \times \dot{\overline{x}}^i\, \overline{\mathbf{g}}_i$ is the Coriolis acceleration,
- $\dot{\mathbf{\Omega}} \times (\overline{x}^i\, \overline{\mathbf{g}}_i)$ is acceleration due to spin up/down of $\overline{O}$,
- $\mathbf{\Omega} \times (\mathbf{\Omega} \times \overline{x}^i\, \overline{\mathbf{g}}_i)$ is the centrifugal force.

The three last accelerations are caused by rotation or change in rotation of $\overline{O}$.

In rotating cylinder coordinates $\{\overline{\mathbf{e}}_R, \overline{\mathbf{e}}_{\overline{\theta}}, \mathbf{e}_z\}$ with coordinates $(R, \overline{\theta}, z)$, where $\overline{\theta} = \theta - \Omega$, the terms in (11.8) become

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \overline{x}^i\, \overline{\mathbf{g}}_i) = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = -\Omega^2 R \overline{\mathbf{e}}_R \tag{11.9}$$

$$\dot{\boldsymbol{\Omega}} \times (\overline{x}^i\, \overline{\mathbf{g}}_i) = \dot{\Omega} R \overline{\mathbf{e}}_{\overline{\theta}} \tag{11.10}$$

$$2\boldsymbol{\Omega} \times \overline{v}_i \overline{\mathbf{g}}_i = 2\Omega(\overline{v}_R \overline{\mathbf{e}}_{\overline{\theta}} - \overline{v}_{\overline{\theta}} \overline{\mathbf{e}}_R) \tag{11.11}$$

## 12 Navier Stokes equations in cylinder coordinates

In coordinate free form, the compressible Navier Stokes equations including momentum and mass conservation can be written

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \mu \nabla^2 \mathbf{v} \tag{12.1}$$

and

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{v}) = 0. \tag{12.2}$$

where $\mathbf{v}$ is velocity, $p$ is the pressure, $\rho$ the mass density and $\mu$ is the dynamic viscosity, $\nu = \mu/\rho$ is the kinematic viscosity. It is included in the definition of the Reynolds-number $Re = UL/\nu$.

When deriving the Navier Stokes equations in cylinder coordinates, we start writing them in generalized coordinates $x^i$. Let us express the momentum equation (12.1) selecting the covariant basis $\{\mathbf{g}_i\}$. Note that it is equivalent to express the equations in the $\{\mathbf{g}^i\}$ system. Assuming that the coordinate system does not vary in time, $\partial \mathbf{g}_i/\partial t = \dot{\mathbf{g}}_i = 0$, then

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial v^i}{\partial t}\mathbf{g}_i.$$

The advective term becomes

$$\mathbf{v}\cdot\nabla\mathbf{v} = v^l \mathbf{g}_l \cdot (v^i{}_{|k})\mathbf{g}_i \mathbf{g}^k = v^l (v^i{}_{|k})\mathbf{g}_i \delta^k_l = v^k (v^i{}_{|k})\mathbf{g}_i.$$

For the pressure gradient, we have $\nabla p = p^{|i}\mathbf{g}_i$, and we have previously shown that

$$\nabla^2 \mathbf{v} = (g^{kl} v^i{}_{|l})_{|k} \mathbf{g}_i = v^i|^k_k \mathbf{g}_i.$$

Substituting these terms into (12.1), we obtain the Navier-Stokes equation for generalized coordinates

$$\rho\left(\frac{\partial v^i}{\partial t} + v^k v^i{}_{|k}\right) = -p^{|i} + \mu v^i|^k_k. \tag{12.3}$$

The equation of continuity becomes

$$\frac{\partial \rho}{\partial t} + (\rho v^i)_{|i} = 0. \tag{12.4}$$

The advection term and the pressure term can be written

$$v^k v^i{}_{|k} = v^k (v^i{}_{,k} + v^\sigma \{{}_\sigma{}^i{}_k\}) \quad \text{and} \quad p^{|i} = g^{ik} p_{,k}.$$

Introducing cylinder coordinates and physical components of the velocity, the equation of radial momentum balance becomes

$$\frac{\partial v_R}{\partial t} + v_R v_{R,R} + \frac{v_\theta v_{R,\theta}}{R} + v_z v_{R,z} - \frac{v_\theta^2}{R} = \\ -\frac{p_{,R}}{\rho} + \frac{\nu}{R}\left((Rv_{R,R})_{,R} + \frac{v_{R,\theta\theta}}{R} + Rv_{R,zz} - \frac{v_R}{R} - \frac{2v_{\theta,\theta}}{R}\right)$$

The momentum balance equations in the azimuthal and axial directions can be derived in the same way. We do not write them here.

## 12.1 The momentum equations in rotating cylinder coordinates

An extensive treatment of rotating flows and the basic equations of rotating flows is given by Childs, see ([2]). Navier Stokes equations in rotating cylinder coordinates is derived by using the results obtained in section (11). We add the centrifugal and Coriolis terms to the equations in stationary cylinder coordinates. The radial momemtum equation in the rotating frame become

$$\frac{\partial v_R}{\partial t} + v_R v_{R,R} + \frac{v_\theta v_{R,\theta}}{R} + v_z v_{R,z} - \frac{v_\theta^2}{R} - \Omega^2 R - 2\Omega v_\theta = \\ -\frac{p_{,R}}{\rho} + \frac{\nu}{R}\left((Rv_{R,R})_{,R} + \frac{v_{R,\theta\theta}}{R} + Rv_{R,zz} - \frac{v_R}{R} - \frac{2v_{\theta,\theta}}{R}\right)$$

## 12.2 The stress tensor

The physical components of the stress tensor $\sigma_{ik} = -p\delta_{ik} + 2\eta S_{ik}$ can be written in cylinder coordinates using the physical components of the strain rate tensor (16.4), we get

$$\begin{aligned} \sigma_{RR} &= -p + 2\mu u_{R,R} \\ \sigma_{R\theta} &= \mu(u_{\theta,R} - u_\theta/R + u_{R,\theta}/R) \\ \sigma_{\theta\theta} &= -p + (2\mu/R)(u_{\theta,\theta} + u_R) \\ \sigma_{\theta z} &= \mu(u_{z,\theta}/R + u_{\theta,z}) \\ \sigma_{zz} &= -p + 2\mu u_{z,z} \\ \sigma_{zR} &= \mu(u_{R,z} + u_{z,R}) \end{aligned} \tag{12.5}$$

In the incompressible case ${v^i}_{|i} = 0$ and $C \to \infty$. We may write

$$p|_i^i = -\rho(v^i v^k)_{|k|i} \tag{12.6}$$

Notice that in the incompressible case

$$(v^i v^k)_{|k|i} = {v^i}_{|k|i} v^k + {v^i}_{|k} {v^k}_{|i}, \tag{12.7}$$

since generally ${v^i}_{|k|i} \neq {v^i}_{|i|k} = 0$. In the incompressible case with Cartesian coordinates differentiation commute and

$$\frac{\partial^2 p}{\partial x^i \partial x^i} = -\frac{\partial v^i}{\partial x^j}\frac{\partial v^j}{\partial x^i} = \frac{1}{4}\left(\frac{\partial v^j}{\partial x^i} - \frac{\partial v^i}{\partial x^j}\right)\left(\frac{\partial v^j}{\partial x^i} - \frac{\partial v^i}{\partial x^j}\right) - \frac{1}{4}\left(\frac{\partial v^j}{\partial x^i} + \frac{\partial v^i}{\partial x^j}\right)\left(\frac{\partial v^j}{\partial x^i} + \frac{\partial v^i}{\partial x^j}\right) = \Omega_{ij}\Omega_{ij} - S_{ij}S_{ij},$$

where $S_{ij}$ is the strain rate tensor and $\Omega_{ij} = \omega_{ij}/2$ and $\omega_{ij}$ is rotation tensor as defined in section (10). This does not generally apply since the second covariant derivatives do not commute, i.e. ${v^i}_{|k|i} \neq {v^i}_{|i|k}$ in curved space.

We recognize $(\Omega_{ij}\Omega_{ij} - S_{ij}S_{ij})/2 = Q > 0$ as Hunt's criteria to identify a vortex, see [4], where rotation dominates over strain. The $Q$ criteria should be used with some care in the general case since it is not clear that $Q$ as defined above is a tensorial quantity. Generally, $2Q = -\rho(v^i v^k)_{|k|i}$ should be used. This deserves some additional analysis since $S_{ij} = (\partial v^i/\partial x^j + \partial v^j/\partial x^i)/2$ is not a tensor. The criteria for identifying a vortex has also been discussed Jeong and Hussain, see [6], using the second eigenvalue, $\lambda_2 < 0$ criteria of $S_{ij}S_{ij} + \Omega_{ij}\Omega_{ij}$. As proposed in [6], it is not clear that $\lambda_2$ is a tensorial quantity. It should be used with care in non-Cartesian coordinate systems and deserves some additional analysis.

## 13 Lighthill's equation in generalized coordinates

Lighthill's equation can be derived from the Navier-Stokes equations, involving both thermal and viscous dissipation, see (James Lighthill). Here we neglect effects of dissipation and start with the Euler equations for compressible flows

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p, \qquad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0. \tag{13.1}$$

By assuming that the motion is adiabatic, i.e isentropic flow, $\delta p = C^2\delta\rho$, where $p$ is the pressure, $C$ is the sound speed and $\rho$ is the density, one can show that the equations can be written in an equivalent form where the left hand side has the form of a wave-equation. In coordinate-free form, Lighthill's equation can be written

$$\frac{1}{C^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \nabla\cdot(\nabla\cdot(\rho\mathbf{v}\mathbf{v}))$$

where $\rho\mathbf{v}\mathbf{v}$ is the momentum flux density tensor.

From the Euler equations (13.1) we have

$$0 = \frac{\partial(\rho v^i)}{\partial t} - \rho\frac{\partial v^i}{\partial t} - v^i\frac{\partial \rho}{\partial t} = \frac{\partial(\rho v^i)}{\partial t} + (\rho v^k v^i)_{|k} + p^{|i}.$$

Since $g^{ij}{}_{|k} = 0$, and lowering the index, we can write $p^{|i} = g^{ik} p_{|k} = (pg^{ik})_{|k}$ and

$$\frac{\partial(\rho v^i)}{\partial t} + (\rho v^i v^k + pg^{ik})_{|k} = 0.$$

Assuming isentropic conditions and constant speed of sound $C$, $\delta p = C^2 \delta \rho$ and we have

$$\frac{1}{C^2}\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 \rho}{\partial t^2}.$$

Taking the time derivative of the equation of continuity and the divergence of the momentum equation (13.1) we get

$$\frac{\partial^2 \rho}{\partial t^2} = -\frac{\partial(\rho v^i)_{|i}}{\partial t} = (\rho v^i v^k + pg^{ik})_{|k|i}$$

which can be written

$$\frac{1}{C^2}\frac{\partial^2 p}{\partial t^2} - p|_i^i = (\rho v^i v^k)_{|k|i}. \tag{13.2}$$

This is the component form of Lighthill's equation in generalized coordinates.

## 14 Lighthill's equation in cylinder coordinates

Now we want to express the right side of equation (13.2) in cylinder coordinates with the physical velocity components $(v_R, v_\theta, v_z)$ as arguments. The term $p|_i^i$ is obtained in section (9), so we skip it here. The double divergence is more complex. Let us take it in steps, first we write

$$(\rho v^i v^k)_{|k|i} = T^{ik}{}_{|k|i} = \left(\frac{1}{\sqrt{g}}(\sqrt{g}T^{ik})_{,k} + T^{lk}\{{}_l{}^i{}_k\}\right)_{|i} = a^i{}_{|i}.$$

The components of vector $\mathbf{a}$ are

$$a^i = T^{ik}{}_{|k} = T^{ik}{}_{,k} + T^{lk}\{{}_l{}^i{}_k\} + T^{il}\{{}_k{}^k{}_l\}.$$

The divergence of $a^i$ becomes

$$a^i{}_{|i} = \frac{1}{\sqrt{g}}\left(\sqrt{g}a^i\right)_{,i} = \frac{a^1}{R} + a^1_{,R} + a^2_{,\theta} + a^3_{,z}.$$

By substituting the expression $a^i = T^{ik}{}_{|k}$ into the above relation yields

$$\begin{aligned} T^{ik}{}_{|k|i} \quad = a^i{}_{|i} = \quad & \tfrac{1}{R}\left(T^{1k}{}_{,k} - RT^{22} + \tfrac{1}{R}T^{11}\right) \\ & + \left(T^{1k}{}_{,k} - RT^{22} + \tfrac{1}{R}T^{11}\right)_{,R} \\ & + \left(T^{2k}{}_{,k} + \tfrac{1}{R}(T^{12} + T^{21}) + \tfrac{1}{R}T^{21}\right)_{,\theta} \\ & + \left(T^{3k}{}_{,k} + \tfrac{1}{R}T^{31}\right)_{,z}. \end{aligned}$$

Since $T^{ik}$ is symmetric, this expression can be expanded and simplified

$$\begin{aligned} T^{ik}{}_{|k|i} \quad & = T^{11}{}_{,RR} + \tfrac{2}{R}T^{11}{}_{,R} + T^{22}{}_{,\theta\theta} - 2T^{22} + T^{33}{}_{,zz} + \tfrac{4}{R}T^{12}{}_{,\theta} \\ & + \tfrac{2}{R}T^{13}{}_{,z} + 2T^{12}{}_{,R\theta} + 2T^{23}{}_{,\theta z} + 2T^{13}{}_{,Rz} - RT^{22}{}_{,R}. \end{aligned}$$

Substituting the phyaical components of the velocity field, we have

$$
(T^{ij}) = \begin{pmatrix} \rho(v^1)^2 & \rho v^1 v^2 & \rho v^1 v^3 \\ - & \rho(v^2)^2 & \rho v^2 v^3 \\ - & - & \rho v^3 v^3 \end{pmatrix} = \begin{pmatrix} \rho v_R^2 & \rho v_R v_\theta / R & \rho v_R v_z \\ - & \rho v_\theta^2 / R^2 & \rho v_\theta v_z / R \\ - & - & \rho v_z^2 \end{pmatrix},
$$

which implies

$$
\begin{aligned}
T^{ik}{}_{|k|i} &= (\rho v_R^2)_{,RR} + (\rho v_\theta^2/R^2)_{,\theta\theta} + (\rho v_z^2)_{,zz} + \tfrac{2}{R}(\rho v_R^2)_{,R} \\
&+ \tfrac{2}{R^2}(\rho v_R v_\theta)_{,\theta} + \tfrac{2}{R}(\rho v_R v_z)_{,z} + \tfrac{2}{R}(\rho v_R v_\theta)_{,R\theta} + \tfrac{2}{R}(\rho v_\theta v_z)_{,\theta z} \\
&+ 2(\rho v_R v_z)_{,Rz} - \tfrac{1}{R}(\rho v_\theta^2)_{,R},
\end{aligned} \tag{14.1}
$$

which is the right hand side of Lighthill's equation in cylinder coordinates with the physical components of the velocity field as arguments. Notice that $T^{ik}{}_{|k|i}$ is a scalar.

## 15 The RANS equations

A field quantity $f$ is decomposed in an average quantity $F$ and a fluctuating quantity $f'$. We write $f = F + f'$. The averaging procedures are such that

$$F = \overline{f} \quad \Rightarrow \tag{15.1}$$

$$\overline{f'} = 0, \tag{15.2}$$

$$\overline{\overline{f}g} = \overline{f}\,\overline{g}, \tag{15.3}$$

$$\overline{\overline{f}g'} = 0. \tag{15.4}$$

For details, see ([13]) where the evolution equation for the Reynolds stress is developed in Cartesian coordinates. An extension to generalized coordinates follows below.

Assuming incompressibility $v^i{}_{|i} = 0$, both $v'^i{}_{|i}$ and $V^i_{|i} = 0$. Taking equation (12.3), substituting the decomposition of $f$ and averaging, we get the averaged momentum equation

$$\frac{\partial V^i}{\partial t} + V^k V^i_{|k} = -\frac{1}{\rho} P^{|i} + \nu V^i|^k_k - R^{ik}{}_{|k}. \tag{15.5}$$

The terms $R^{ik} = \overline{v'^i v'^k}$ represent the contravariant components of the Reynolds stress tensor.

Subtracting (15.5) from the equation for $v^i$ gives the following equation

$$\frac{\partial v'^i}{\partial t} + V^k v'^i{}_{|k} + v'^k V^i_{|k} = -\frac{1}{\rho} p'^{|i} - (v'^i v'^k - \overline{v'^i v'^k})_{|k} + \nu v'^i|^k_k, \tag{15.6}$$

which is the fluctuating momentum equation. From now, we write $x$ for the fluctiating quantities instead of $x'$. Taking the $i$ component the fluctuating momentum equation and multiplying with $v^j$, the $j$ component of the same equation and multiplying with $v^i$ summing and averaging, we get the Reynolds stress evolution equation in generalized coordinates

$$\begin{array}{rll}\frac{\partial(\overline{v^i v^j})}{\partial t} = & -V^k(\overline{v^i v^j})_{|k} & \text{convection} \\ & -(\overline{v^j v^k}V^i_{\ |k} + \overline{v^i v^k}V^j_{\ |k}) & \text{turbulent production} \\ & -(\overline{u^i u^j u^k})_{|k} & \text{turbulent diffusion} \\ & -\frac{1}{\rho}(g^{i\sigma}(\overline{pv})^j_{\ |\sigma} + g^{j\sigma}(\overline{pv})^i_{\ |\sigma}) & \text{pressure diffusion} \\ & +\frac{1}{\rho}(g^{i\sigma}\overline{pv^j_{\ |\sigma}} + g^{j\sigma}\overline{pv^i_{\ |\sigma}}) & \text{pressure strain} \\ & +\nu g^{k\sigma}(\overline{v^i v^j})_{|k\sigma} & \text{viscous diffusion} \\ & -\nu g^{k\sigma}(\overline{v^i_{\ |k}v^j_{\ |\sigma}} + \overline{v^i_{\ |\sigma}v^j_{\ |k}}) & \text{viscous dissipation}\end{array} \tag{15.7}$$

Notice the existense of a term $\overline{u^i u^j u^k}$ which implies a closure problem. The terms above can be written out in cylinder coordinates. Note that $\overline{v^i v^j}$ is a contravariant tensor.

We write the evolution equation for the this expression, the Reynolds stress (15.7) as follws

$$\frac{\partial R^{ij}}{\partial t} = C^{ij} + T_p^{ij} + T_d^{ij} + P_d^{ij} + P_s^{ij} + V_{diff}^{ij} + V_{diss}^{ij}. \tag{15.8}$$

This tensors physcal components can be calculated using expression (7.8). For example $A_{r\theta} = rA^{12}, A_{rz} = A^{13}, A_{\theta\theta} = r^2 A^{22}$. Let us as calculate some of the terms in cylinder coordinates as examples. The pressure diffusion tensor is

$$P_d^{ij} = -\frac{1}{\rho}(g^{i\sigma}(\overline{pv})^j_{|\sigma} + g^{j\sigma}(\overline{pv})^i_{|\sigma}).$$

Substituting $A^i = (\overline{pv})^i = \overline{pv^i}$, we get using (6.6)

$$P_d^{ij} = -\frac{1}{\rho}\left(g^{i\sigma}\left(\frac{\partial A^j}{\partial x^\sigma} + A^s\{_s{}^j{}_\sigma\}\right) + g^{j\sigma}\left(\frac{\partial A^i}{\partial x^\sigma} + A^s\{_s{}^i{}_\sigma\}\right)\right).$$

Using this formula, the fact that $g^{ij}$ is diagonal and given by (7.4), that all Christoffel symbols $\{_i{}^j{}_k\}$ are zero except $\{_1{}^2{}_2\} = \{_2{}^2{}_1\} = 1/r, \{_2{}^1{}_2\} = -r$ and substituting for the physical vector components of $A^i$, we get the following physical components for the pressure diffusion tensor

$$\begin{aligned}
P_d(rr) &= P_d^{11} = -\frac{2}{\rho}\frac{\partial\overline{pv_r}}{\partial r} = -\frac{2}{\rho r}\left(\frac{\partial r\overline{pv_r}}{\partial r} - \overline{pv_r}\right), \\
P_d(r\theta) &= rP_d^{12} = -\frac{1}{\rho r}\left(\frac{\partial r\overline{pv_\theta}}{\partial r} + \frac{\partial\overline{pv_r}}{\partial\theta} - 2\overline{pv_\theta}\right), \\
P_d(rz) &= P_d^{13} = -\frac{1}{\rho}\left(\frac{1}{r}\frac{\partial r\overline{pv_z}}{\partial r} - \frac{1}{r}\overline{pv_z} + \frac{\partial\overline{pv_r}}{\partial z}\right), \\
P_d(\theta\theta) &= r^2 P_d^{22} = -\frac{2}{\rho r}\left(\frac{\partial\overline{pv_\theta}}{\partial\theta} + \overline{pv_r}\right), \\
P_d(\theta z) &= rP_d^{23} = -\frac{1}{\rho}\left(\frac{1}{r}\frac{\partial\overline{pv_z}}{\partial\theta} + \frac{\partial\overline{pv_\theta}}{\partial z}\right), \\
P_d(zz) &= P_d^{33} = -\frac{2}{\rho}\frac{\partial\overline{pv_z}}{\partial z}.
\end{aligned}$$

# 16 Basic equations from the theory of elasticity

The treatment here is a generalization of the elasticity equations given in ([8]). Consider a point $\mathcal{P}$ in a solid and assume the body is deformed such that point $\mathcal{P}$ is displaced to another point $\mathcal{P}'$. The displacement vector is $\mathbf{u}_p = \mathbf{r}'_p - \mathbf{r}_p$. Consider another point $\mathcal{Q}$ close to $\mathcal{P}$. After deformation $\mathcal{Q}$ is moved to $\mathcal{Q}'$. The displacement vector for $\mathcal{Q}$ is $\mathbf{u}_q = \mathbf{r}'_q - \mathbf{r}_q$. The change in the vector connecting $\mathcal{P}$ and $\mathcal{Q}$ due to displacement is

$$d\mathbf{u} = \mathbf{u}_q - \mathbf{u}_p = (\mathbf{r}'_q - \mathbf{r}'_p) - (\mathbf{r}_q - \mathbf{r}_p),$$

which on component form is written

$$du_i = dx'_i - dx_i.$$

The distance between $\mathcal{P}$ and $\mathcal{Q}$ before displacement is $dl^2 = dx_i dx_i$, and after displacement is $dl'^2 = dx'_i dx'_i$. Due to continuity we write

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j.$$

Then

$$dl'^2 = dx_i^2 + 2u_{ij} dx_i dx_j,$$

where

$$u_{ij} = \frac{1}{2}\left\{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right\}. \tag{16.1}$$

Let us show that $u_{ij}$ is indeed a tensor we consider two coordinate systems $\mathcal{O}$ and $\overline{\mathcal{O}}$. The line elements $dl'$ and $dl$ are invariants. We write

$$(dl')^2 = dl^2 + 2u_{ij} dx^i dx^j,$$
$$(d\bar{l}')^2 = d\bar{l}^2 + 2\overline{u}_{ij} d\overline{x}^i d\overline{x}^j.$$

Since $dx^i = (\partial x^i/\partial \overline{x}^j) d\overline{x}^j$ and the line elements are invariants, we get

$$\overline{u}_{ij} = \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j} u_{kl},$$

showing that $u_{ij}$ are the covariant tensor components. The expression for $u_{ij}$ given in (16.1) is valid for Cartesian coordinates. Generally we may write the strain tensor as

$$2u_{ij} = ((u_{i|j} + u_{j|i}) + u_{k|j} u_{k|i}). \tag{16.2}$$

If the deformation is small, we may omit the non-linear term in the above expression, then we may write

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right) - u_\sigma \{{}_i{}^\sigma{}_j\}. \tag{16.3}$$

Recall that the physical components of a vector $v(\alpha) = v^\alpha\sqrt{g_{\alpha\alpha}}$ and for a tensor $T(\alpha,\beta) = T^{\alpha\beta}\sqrt{g_{\alpha\alpha}g_{\beta\beta}}$ etc... Since a symmetric tensor is symmetric in all coordinate systems, we write for $i,j = 1,2,3$

$$u_{ij} = \begin{pmatrix} u_{1,1} & (u_{1,2}+u_{2,1})/2 - u_2/r & (u_{1,3}+u_{3,1})/2 \\ - & u_{2,2}+ru_1 & (u_{2,3}+u_{3,2})/2 \\ - & - & u_{3,3} \end{pmatrix}.$$

Using equation (7.7) for the physical components of a vector we calculate the physical components of the strain tensor which has the same form as the strain rate tensor, see (10.3)

$$u(ij) = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2r}\left(\frac{\partial u_r}{\partial\theta} + r\frac{\partial u_\theta}{\partial r} - u_\theta\right) & \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\ - & \frac{1}{r}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right) & \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial\theta}\right) \\ - & - & \frac{\partial u_z}{\partial z} \end{pmatrix}. \tag{16.4}$$

The stress tensor and body forces.
The total force on a body is $\int \mathbf{F}dV$, where $\mathbf{F}$ is force/volume. According to the divergence theorem of integration, for the vector field $\mathbf{a}$

$$\oint_S \mathbf{a}\cdot d\mathbf{s} = \int_V \nabla\cdot\mathbf{a}\,dV.$$

This can be generalized to a tensor field, say $\sigma$

$$\oint_S \sigma_{ik}dS_k = \int_V \frac{\partial\sigma_{ij}}{\partial x_k}dV = \int_V F_i dV.$$

Here $\sigma_{ij}$ is the stress tensor and $F_i$ is the $i$ component of the force (force/volume) related to the stress,

$$F_i = \frac{\partial\sigma_{ik}}{\partial x_k}.$$

This formula is valid in Cartesian coordinates. Generally we may write in coordinate free form

$$\mathbf{F} = \nabla\cdot\sigma.$$

As an example we can derive the co-variant components of the force, $F_i$. Then we may look for expressions like $\sigma_i{}^k{}_{|k}$. We have

$$\nabla\cdot\sigma = \mathbf{g}^k\cdot\frac{\partial\sigma_i{}^j\mathbf{g}^i\mathbf{g}_j}{\partial x^k}.$$

Carrying out the derivation in this expression gives

$$\nabla\cdot\sigma = \mathbf{g}^k\cdot\frac{\partial\sigma_i{}^j}{\partial x^k}\mathbf{g}^i\mathbf{g}_j + \sigma_i{}^j\mathbf{g}^k\cdot\frac{\partial\mathbf{g}^i}{\partial x^k}\mathbf{g}_j + \sigma_i{}^j\mathbf{g}^k\cdot\mathbf{g}^i\frac{\partial\mathbf{g}_j}{\partial x^k} = \delta_j^k\frac{\partial\sigma_i{}^j}{\partial x^k}\mathbf{g}^i - \sigma_i{}^j\delta_j^k\{_m{}^i{}_k\}\mathbf{g}^m + \sigma_i{}^j\delta_m^k\{_j{}^m{}_k\}\mathbf{g}^i,$$

where we have used (4.6,4.7). Cleaning up, we get

$$F_i = (\nabla\cdot\sigma)_i = \sigma_i{}^k{}_{|k} = \frac{\partial\sigma_i{}^k}{\partial x^k} - \sigma_s{}^k\{_i{}^s{}_k\} + \sigma_i{}^s\{_s{}^k{}_k\}.$$

As an example of equivalence of the co and contravariant vectors, we can use the contravariant components which are more convenient when calculating the physical components. We have

$$F^i = \sigma^{ik}{}_{|k} = \frac{\partial \sigma^{ik}}{\partial x^k} + \sigma^{sk}\{{}_s{}^i{}_k\} + \sigma^{is}\{{}_s{}^k{}_k\},$$

which implies that

$$F(\alpha) = \sqrt{g_{\alpha\alpha}}\Big(\frac{\partial \sigma^{\alpha k}}{\partial x^k} + \sigma^{sk}\{{}_s{}^\alpha{}_k\} + \sigma^{\alpha s}\{{}_s{}^k{}_k\}\Big). \tag{16.5}$$

Introducing physical components, we get

$$F_r = \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}\Big(\sigma_{rr} - \sigma_{\theta\theta}\Big), \tag{16.6}$$

$$F_\theta = \frac{\partial \sigma r\theta}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta}, \tag{16.7}$$

$$F_z = \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r}. \tag{16.8}$$

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