FFI RAPPORT

OPTIMISATION OF FLARE AND CHAFF PROGRAMS - An analytical approach

HOVLAND Harald

FFI/RAPPORT-2006/01460

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FORSVARETS FORSKNINGSINSTITUTT
Norwegian Defence Research Establishment
P O Box 25, NO-2027 Kjeller, Norway

FORSVARETS FORSKNINGSINSTITUTT (FFI)
Norwegian Defence Research Establishment

P O BOX 25
N0-2027 KJELLER, NORWAY

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE
(when data entered)

REPORT DOCUMENTATION PAGE

1) PUBL/REPORT NUMBER FFI/RAPPORT-2006/01460	2) SECURITY CLASSIFICATION UNCLASSIFIED	3) NUMBER OF PAGES 65
1a) PROJECT REFERENCE FFI-II/1008/915	2a) DECLASSIFICATION/DOWNGRADING SCHEDULE -	

4) TITLE
OPTIMISATION OF FLARE AND CHAFF PROGRAMS - An analytical approach

5) NAMES OF AUTHOR(S) IN FULL (surname first)
HOVLAND Harald

6) DISTRIBUTION STATEMENT
Approved for public release. Distribution unlimited. (Offentlig tilgjengelig)

7) INDEXING TERMS

IN ENGLISH:	IN NORWEGIAN:
a) Countermeasures	a) Motmidler
b) Flare	b) Flare
c) Chaff	c) Chaff
d) Optimisation	d) Optimalisering
e) Missile approach warning system	e) Missilvarsler

THESAURUS REFERENCE:

8) ABSTRACT

A statistical model is presented that expresses the mission survival probability of an aircraft as a function of various input parameters such as missile attack rate, false alarm rate, missile detection probability and mission duration. Use of the model enables a dynamic optimisation for efficient use of expendable countermeasures. It also enables optimisation of equipment procurement by quantifying the effect of improving the performance of specific components. The model also results in a method for dynamically optimising the key parameters of a missile approach warning system during the mission, ideally leading to a significant increase in the aircraft protection.

9) DATE	AUTHORIZED BY This page only	POSITION
2006-05-09	Vidar S Andersen	Director

ISBN 82-464-1005-9

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE
(when data entered)

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OPTIMISATION OF FLARE AND CHAFF PROGRAMS - An analytical approach

1 INTRODUCTION

1.1 Background

This work has been performed as a part of the FFI project 1008 Operational EW support of the Armed Forces. The main project target is enhancing the Defensive Aids Suite (DAS) of Royal Norwegian Air Forces (RNoAF) platforms for the protection against infrared (IR) and radar guided missile threats. The basis of the report is a problem related to the IR threat, but, as the report will show, the results are directly transferable to the radar threat. In principle, the model developed in this work is applicable to any platforms (air, sea, land) deploying resource limited expendables, such as flare, chaff, smoke or even hard kill units.

1.2 Problem framework

An aircraft sent on a mission can be attacked by IR or radar homing missiles. One self protection measure is to deploy flares or chaff to prevent incoming missiles to lock on their target. These countermeasures are deployed from a number of dispensers located on the airframe. The dispenser capacity is typically limited to 15-30 cartridges, and there are typically between 2 and 10 dispensers on an aircraft. Because the resulting number of countermeasure cartridges is relatively limited, they are mostly deployed reactively, after an incoming threat has been detected. The threat detection can be done either manually, or automatically by using a missile approach warning system (MAWS). The MAWS must be able to discriminate between noise/false targets and real incoming threats, while being sensitive enough to detect the target.

1.3 Problem statement

The problem to be dealt with in this work can be stated as follows:

> *Given that an aircraft has a limited amount of flares available during a mission, what is the optimum flare program size, given that the flare program efficiency depends on the number of flares used in the program.*

Normally, an increase in flare program size will increase the program efficiency, although this may not be the case. On the other hand, high flare consumption per program will increase the risk of spending all the flares. In that case, one might be forced to meet subsequent missile attacks without the use of flares.

The main factors related to this problem are the following:

1. The choice of missile attack probability model, and the consequences of this choice
2. The probability of surviving a missile attack as a function of the number of flares used in a flare program
3. The probability of surviving a missile attack, given that there are no flares left
4. The probability of false alarms, reducing the amount of flares available
5. Expected mission time (or remaining time if the mission has started)
6. Uncertainty in input parameters, and the resulting consequences

1.4 Scope of this work

The main goal of this work is **not** to determine the aircraft survival probability during a single, specified scenario with given parameters, but to study how the various parameters affects the choice of flare program size. The model resulting from this work can nevertheless be applied to specific scenarios.

The driving force behind the work is the wish for a comprehensive, quantitative model for EW self protection. Inquiries to several leading members of the EW community has neither revealed nor indicated the existence of such a model. Alternative approaches have either been of the Monte-Carlo simulation type, or a functional analysis approach, in which the goal is to find the requirements of what is needed to decoy a specific set of missiles. The first approach is easy to implement, but does not easily provide insight into the issues. The second approach answers certain central questions, but does not give any information of countermeasures consumption. As will be seen in this report, there are cases where flare programs are so big that the depletion effect means that they will not be competitive with smaller programs, regardless of their efficiency. They do not answer the question of whether or not a second program deployed just after the first one will be beneficial or not. The model being developed in this work provides additional information about these issues, can contribute to enhanced self protection, and equally important, provides a measure for this.

In the first part of this report a method is introduced in which the statistical probability of mission survival is determined. The model is first presented in a constrained and idealised scenario in which an idealised MAWS is used, and thereafter the model is extended to gradually become more general and realistic. Thereafter various parameters are studied, as well as their effect on the model properties. Finally, the model is applied to some numerical scenarios, and the results are compared with numerical simulations.

1.5 Report availability

This report is also available on the internet: http://rapporter.ffi.no/rapporter/2006/01460.pdf.

2 BASIC MODEL

In this chapter, the following assumptions are made:

1. All missile attacks are considered to be independent events.
2. There are a large number of missiles present in the scenario. Not all missiles need to be fired; they only need to be potential threats.
3. There is a constant missile attack probability throughout the mission.
4. There is a constant probability of surviving an attack, given the use of a specified flare programme.
5. The total number of flare cartridges left in the dispensers is divisible by the program flare size at all times.
6. The MAWS is ideal, without any false alarms.

We now define the following parameters:

N:	Total (remaining) flare capacity.
n:	Number of flares in a salvo (program).
S:	Number of (remaining) salvos available. In the simplified model this is given by $S=N/n$.
P:	Mission survival probability.
$P_{Survive,i}$:	The probability of surviving i missile attacks.
$p_{MA,j}(t)$:	The probability for j different attacks to take place within the time t (from t_0=0).
γ_{MA}:	Expected number of missile attacks per unit time, for simplicity termed missile attack rate rate in this report.
ρ_n:	Probability of surviving a missile attack when an n-flare salvo is deployed.

The last definition can be generalised to the value n=0, when no flares are available during an attack.

The probability *P* to survive a mission will in general be given as the probability for a sequence of events to take place times the probability of surviving this sequence of events, summed up over all possible sequences of events.

One way to express this is:

P= (Probability to encounter 0 attacks) × (Probability to survive 0 attacks)
+ (Probability to encounter 1 attack) × (Probability to survive 1 attack)
+ (Probability to encounter 2 attacks) × (Probability to survive 2 attacks)
+ ......

summed up for all possible number of attacks (from 0 to infinity). This will be the approach used in the basic model. Mathematically, this can be written:

$$P = \sum_{i=0}^{\infty} p_{MA,i}(t) P_{Survive,i}, \tag{2.1}$$

where *i* is the number of attacks. Given the assumptions in the beginning of chapter 2, the probability $p_i(t)$ of *i* attacks within a time interval of duration *t* will have a Poisson distribution, given by:

$$p_{MA,i}(t) = \frac{e^{-\gamma_{MA} t} (\gamma_{MA} t)^i}{i!} \tag{2.2}$$

The Poisson distribution for different values of the number of attacks *i* is given as a function of time *t* in Figure 2.1.

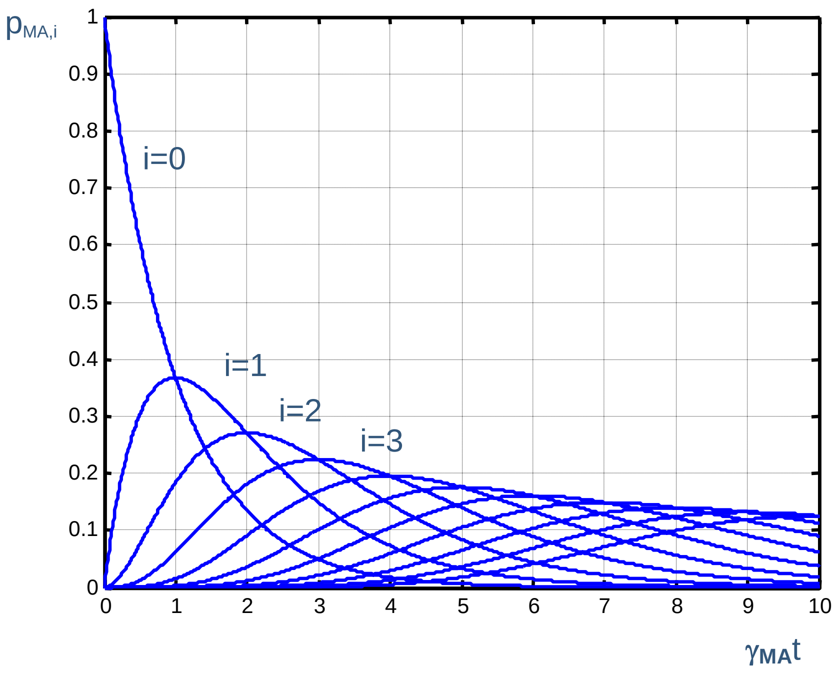

Figure 2.1 Poisson distribution as a function of time×rate, for different values of the number of attacks i. Note that this is a somewhat unconventional way to present the Poisson distribution. (In general, it is presented as a function of the number of attacks i, with a fixed time.)

As there has been between zero and an infinite number of attacks between time 0 and *t*, we have:

$$\sum_{i=0}^{\infty} p_{MA,i}(t) = 1 \tag{2.3}$$

The probability $P_{survive,i}$ of surviving *i* attacks is given by:

$$P_{survive,i} = \begin{cases} \rho_n{}^i & i \le S \\ \left(\frac{\rho_n}{\rho_0}\right)^S \rho_0{}^i & i > S \end{cases} \tag{2.4}$$

The missile attack survival probability ρ_n is given as the sum of the probability of the missile not working (= survival probability ρ_0 of an attack when countermeasures are not deployed) and the probability that the missile functions correctly (= 1-ρ_0), but is decoyed effectively by the countermeasures (= flare decoy efficiency $p_{CME,n}$). We then get the expression:

$$\rho_n = \rho_0 + (1 - \rho_0)\, p_{CME,n} \tag{2.5}$$

The two expressions in equation (2.4) are different, depending on whether or not there are enough flares to handle all the attacks.

Summed up over all possible events, we get:

$$\begin{aligned} P &= \sum_{i=0}^{\infty} \frac{e^{-\gamma_{MA}t} (\gamma_{MA}t)^i}{i!} P_{survive,i} \\ &= \sum_{i=0}^{S} \frac{e^{-\gamma_{MA}t} (\gamma_{MA}t)^i}{i!} \rho_n{}^i + \sum_{i=S+1}^{\infty} \frac{e^{-\gamma_{MA}t} (\gamma_{MA}t)^i}{i!} \left(\frac{\rho_n}{\rho_0}\right)^S \rho_0{}^i \\ &= \sum_{i=0}^{S} \frac{e^{-\gamma_{MA}t} (\gamma_{MA}t)^i}{i!} \left(\rho_n{}^i - \left(\frac{\rho_n}{\rho_0}\right)^S \rho_0{}^i\right) + \sum_{i=0}^{\infty} \frac{e^{-\gamma_{MA}t} (\gamma_{MA}t)^i}{i!} \left(\frac{\rho_n}{\rho_0}\right)^S \rho_0{}^i \\ &= e^{-\gamma_{MA}t} \left(\sum_{i=0}^{S} \left(\frac{(\rho_n \gamma_{MA} t)^i - \left(\frac{\rho_n}{\rho_0}\right)^S (\rho_0 \gamma_{MA} t)^i}{i!} \right) + \left(\frac{\rho_n}{\rho_0}\right)^S \sum_{i=0}^{\infty} \frac{(\rho_0 \gamma_{MA} t)^i}{i!} \right) \end{aligned} \tag{2.6}$$

The infinite sum can be simplified, yielding the following expression:

$$P = \left(\frac{\rho_n}{\rho_0}\right)^S e^{(\rho_0 - 1)\gamma_{MA}t} + e^{-\gamma_{MA}t} \sum_{i=0}^{S} \left(\frac{(\rho_n \gamma_{MA} t)^i - \left(\frac{\rho_n}{\rho_0}\right)^S (\rho_0 \gamma_{MA} t)^i}{i!} \right) \tag{2.7}$$

We shall in the following extend the model to make it more realistic.

3 EXTENDED MODEL

3.1 Introduction of a more realistic MAWS and an extended missile model

We will now gradually extend the model developed in chapter 2. We start by replacing assumption 6 by:

> 6 b) The MAWS is not ideal, but has constant false alarm probability per unit time γ_{FA}, in the following termed the false alarm rate.

We also assume the missile detection probability to be p_D.

We now modify the missile attack survival probability to account for the flare decoy efficiency. This modified missile attack survival probability will be called the effective missile attack survival probability. The flare decoy efficiency $p_{CME,n}$ is defined as the probability of the missile being decoyed, given that it otherwise performs flawlessly. This is a parameter that can be measured during classical countermeasures trials, although a weighting as a function of the various missile types present in the scenario should be made.

It is possible to include a non-ideal detection probability in an effective missile attack survival probability ρ'_n using n flares by the following modification:

$$\rho_n \rightarrow \rho'_n = p_D \rho_n + (1 - p_D)\rho_0 \tag{3.1}$$

Here, this probability is given as the sum of probability to detect the missile attack and survive it with the given flare program, and the probability of not detecting the missile attack and survive without the use of countermeasures.

Inserting the expression in equation (2.5) into the expression of the effective missile attack survival probability ρ'_n in equation (3.1), we get:

$$\rho'_n = p_D \left(\rho_0 + (1 - \rho_0)\, p_{CME,n} \right) + (1 - p_D)\rho_0 \tag{3.2}$$

This expression can be simplified into:

$$\rho'_n = \rho_0 + p_D\, p_{CME,n} (1 - \rho_0) \tag{3.3}$$

It should be mentioned that by introducing a detection probability $p_D<1$, an error is introduced in the model. The error is caused by the fact that no countermeasures are consumed if the MAWS does not detect the incoming threat; whereas the model assumes that there is consumption at every missile attack. An artificial over-consumption of flares per unit time is thus introduced, and can be quantified as $(1 - p_D)\gamma_{MA}$. We here choose to ignore this over-consumption, as in all practical cases the consumption is totally dominated by other terms, as

$\gamma_{FA} \gg \gamma_{MA} \gg (1-p_D)\gamma_{MA}$. In the special case where a high missile attack rate is combined with a poorly performing missile warning system, the model would in principle have to be modified, but then one may argue that in such a scenario, countermeasures consumption will not be the biggest problem (missile warning will), and also this is definitely the wrong equipment in the wrong scenario.

Regarding the introduction of false alarms, this represents a significant complication of the model, as we now have two processes (false alarms and missile attacks) competing for a limited number of flares. By nature, the processes have major similarities. Among other, the probability $p_{FA,j}(t)$ to have j false alarms within the time interval t also has a Poisson distribution:

$$p_{FA,j}(t) = \frac{e^{-\gamma_{FA} t} (\gamma_{FA} t)^j}{j!} \qquad (3.4)$$

Given that a false alarm is an event without any attack, the survival probability of such an event equals 1.

We will in the following assume these two processes to be independent, an assumption that is slightly inaccurate. False alarms in the MAWS are generally declared as a result of noise that is uncorrelated with the missile attack, but by definition, any alarm when a missile is incoming will be a true alarm. The two processes can not coinside, and will therefore, mathematically speaking not be independent. If, however, the time intervals are made arbitrarily small, the probability of having coinciding events will be arbitrarily small. This again is nevertheless just a mathematical trick, because a missile attack has an extended duration, typically 3-10 seconds. Compared to the total mission duration, which typically extends over several hours, the error introduced is marginal. This argument is assumed to be of general validity in the model.

The mission survival probability P is again given as the product of the probability of a sequence of events to happen and the probability of surviving this sequence of events, summed up over all possible sequences of events. We must now find an expression for all these events. During a time interval 0...t we have two categories of cases.

- Case 1: There are still flares left at the time t.
- Case 2: All flares have been used at the time t.

Cases of category 1 can occur if the number of missile attacks i is comprised between 0 and S-1, and the number of false alarms j is comprised between 0 and S-1-i. The sum of all salvos fired is then $i + j < S$. The probability $P_{Survive,i}$ of surviving such a case is given by:

$$P_{Survive,i} = \rho'^{\,i}_n \qquad (3.5)$$

The probability for such a case to occur within a time interval 0...t is:

$$p_{FL,i,j}(t) = p_{MA,i}(t)\, p_{FA,j}(t), \text{ where } 0 \le i \le S-1, \text{ and } 0 \le j \le S-1-i \tag{3.6}$$

In the cases of category 2, where all countermeasures have been used at the time *t*, we can define the time t_s at which the last flare salvo was fired. We then have $0 < t_s \le t$. The reason for introducing t_s is that the model underlying the Poisson statistics does not say anything about the positioning of events within the time interval described. Having or not having flares left is of importance to the probability of surviving a missile attack, and hence the sequence of false alarms and missile attacks are of importance if the availability of sufficient amount of countermeasures for both is not specified.

We now define the time interval during which the last flare salvo was fired as dt_s, and assumes this duration to be very short. The reason for introducing a time interval is that the Poisson -statistics are only defined for a time interval. Again, since we do not know when within the interval the event takes place, it is not sufficient to divide the mission in 2 time intervals (before and after t_s), as this does not ensure unambiguousness in the chain of events, when summing up all events. That can only be done by defining a third, very short interval around t_s where the last flare deployment takes place. By summing up for all possible time intervals of length dt_s, one is ensured the uniqueness of all possible events only once. By letting dt_s approach 0 and summing, the sum mathematically becomes an integral.

For an aircraft to survive the mission, it is necessary for it to survive both the attacks during the time interval $0...t_s$ prior to the last countermeasure deployment, the time interval $t_s...t_s+dt_s$ during which the last salvo is deployed, and the last time interval $t_s+dt_s...t$, during which there might also be missile attacks. The total survival probability for such a case equals the product of the survival probability for all three intervals.

Until the time t_s, *S*-1 flare salvos have been fired either due to false alarms or due to missile attacks. The number *i* of missile attacks is then in the range 0,..., *S*-1, and the number of false alarms *j* will be given as $j = S–1–i$. The probability of each of these cases to occur is given by:

$$p_{FL,i,j=S-1-i} = p_{MA,i}(t_s)\, p_{FA,S-1-i}(t_s), \text{ where } 0 \le i \le S-1, \text{ and } j = S-1-i, \tag{3.7}$$

where we have summed over all possible combinations of events.
The survival probability for *i* missile attacks is still given as $P_{survive,i} = \rho'^{\,i}_n$.

During the time interval $t_s...t_s+dt_s$ when the last salvo is fired, two things can happen:

- There is a missile attack, with a probability $\gamma_{MA} dt_s$ and a survival probability ρ'_n.
- There is a false alarm, with a probability $\gamma_{FA} dt_s$ and a survival probability 1.

That this is the expression obtained can be found by looking at the probability of having an event as dt_s approaches 0. We have:

$$p_{MA,1}\left(dt_s\right)=\frac{e^{-\gamma_{MA}dt_s}\gamma_{MA}dt_s}{1}\rightarrow\gamma_{MA}dt_s$$
$$p_{FA,1}\left(dt_s\right)=\frac{e^{-\gamma_{FA}dt_s}\gamma_{FA}dt_s}{1}\rightarrow\gamma_{FA}dt_s \qquad (3.8)$$

because the exponential function tends towards 1. The probability of having multiple events becomes negligible compared to the single event probability when dt_s approaches 0. The probability of these cases to occur is given by:

$$p\left(\text{Probability of a MA to occur during } \mathrm{t_s....t_s}+d\mathrm{t_s}\right)=dt_s\gamma_{MA}$$
$$p\left(\text{Probability of a FA to occur during } \mathrm{t_s....t_s}+d\mathrm{t_s}\right)=dt_s\gamma_{FA} \qquad (3.9)$$

During the last time interval there might be 0 and an infinite amount of missile attacks. There are no flares left, so the probability of each missile attack to occur is given by $\rho'_0 = \rho_0$.
The probability of any one of these last cases to occur is given by:

$$p\left(\text{Probability of k missile attacks during } t_s+dt_s...t\right)=p_k\left(t-t_s\right),\ \text{where k=0,1,...,}\infty \qquad (3.10)$$

It is important to recall that one must, for completeness, sum for all possible values of t_s. This is achieved by summing for all values dt_s between 0 and t. The sum becomes more accurate by letting dt_s tend towards 0; the sum then becomes an integral. The total survival probability P (for both categories) then becomes:

$$P=\left(\sum_{i=0}^{S-1}\sum_{j=0}^{S-1-i}p_{MA,i}\left(t\right)\rho'^{\,i}_n\,p_{FA,j}\left(t\right)\right)$$
$$+\left(\int_0^t\left[\sum_{i=0}^{S-1}\left(p_{MA,i}\left(t_s\right)\rho'^{\,i}_n\,p_{FA,S-1-i}\left(t_s\right)\right)dt_s\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)\sum_{k=0}^{\infty}\left(p_{MA,k}\left(t-t_s\right)\rho_0^{\,k}\right)\right]\right) \qquad (3.11)$$

Inserting the Poisson probability distributions into this expression, we get:

$$P=\sum_{i=0}^{S-1}\left(\frac{e^{-\gamma_{MA}t}\left(\gamma_{MA}t\right)^i}{i!}\rho'^{\,i}_n\sum_{j=0}^{S-1-i}\frac{e^{-\gamma_{FA}t}\left(\gamma_{FA}t\right)^j}{j!}\right)$$
$$+\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)\int_0^t dt_s\sum_{i=0}^{S-1}\left(\frac{e^{-\gamma_{MA}t_s}\left(\gamma_{MA}t_s\right)^i}{i!}\rho'^{\,i}_n\frac{e^{-\gamma_{FA}t_s}\left(\gamma_{FA}t_s\right)^{S-1-i}}{\left(S-1-i\right)!}\right)\sum_{k=0}^{\infty}\frac{e^{-\gamma_{MA}\left(t-t_s\right)}\left(\gamma_{MA}\left(t-t_s\right)\right)^k}{k!}\rho_0^{\,k} \qquad (3.12)$$

This expression can be simplified:

$$P=\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^S e^{(\rho_0-1)\gamma_{MA}t}$$
$$+\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^S e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{u-S}-\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S}\right)\right) \quad (3.13)$$

where

$$\rho'_n=\rho_0+p_D p_{CME,n}\left(1-\rho_0\right)$$

Alternatively, the expression can be written using an integral:

$$P=e^{(\rho'_n-1)\gamma_{MA}t}+\frac{\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^S}{(S-1)!}\int_0^t\left(dt_s t_s^{S-1}\left(e^{(\rho_0-1)\gamma_{MA}t-(\rho_0\gamma_{MA}+\gamma_{FA})t_s}-e^{(\rho'_n-1)\gamma_{MA}t-(\rho'_n\gamma_{MA}+\gamma_{FA})t_s}\right)\right) \quad (3.14)$$
$$\rho'_n=\rho_0+p_D p_{CME,n}\left(1-\rho_0\right)$$

The step by step derivation of these two expressions can be found in appendix A.

3.2 Variability in rates and a possible incomplete last salvo

We will in the following no longer assume that N is divisible by *n*, but rather redefine *S*:

$$S=\text{ceil}\left(N/n\right) \quad (3.15)$$

where the ceil-function rounds up the fraction to the nearest integer. The remaining number of flares available in the last salvo then becomes *r*, given by:

$$r=N-n\left(S-1\right) \quad (3.16)$$

During a mission there can be periods where one *à priori* expects an increased missile attack rate (probability of attack per unit time), for example when entering areas with higher density of air defence. Likewise, it is possible to find areas with an expected increase in the false alarm rate, for example in industrial areas or when flying at high altitudes. Information about this could be determined during mission flight planning.

To see how such an increase in missile attack and false alarm rates may enter the model we can exploit the fact that the rates always occur as a factor together with time in the formulas. An increase of a rate could then be considered as some kind of “time compression”.

We now introduce normalised “intensity times” T_{MA} and T_{FA} for missile attacks and false alarms, respectively. These are defined by their values at *t*=0 and the time derivatives:

$$T_{MA}\left(0\right)=T_{FA}\left(0\right)=0 \quad (3.17)$$

$$\frac{dT_{MA}}{dt} = \frac{\gamma_{MA}(t)}{\gamma_0} \tag{3.18}$$

$$\frac{dT_{FA}}{dt} = \frac{\gamma_{FA}(t)}{\gamma_0} \tag{3.19}$$

where γ_0 is a normalised rate, a rate that can be used as an input parameter in statistical models with Poisson distribution.

As long as one assumes that all events are independent, the missile attack and false alarm probabilities per unit time (denoted rates) are constant in these new normalised time spaces $\mathbb{T}$. The conditions in this new time space are therefore satisfied for continuing using Poisson statistics. The only requirement is that T_{MA} and T_{FA} both be defined for all values of time within the interval 0...*t*. The only requirement necessary for using Poisson statistics is that both γ_{MA} and γ_{FA} are integrable within the required time intervals. Figure 3.1 below shows an example of how the expected average missile attack rate γ_{MA} can be included in the normalised intensity time T_{MA}.

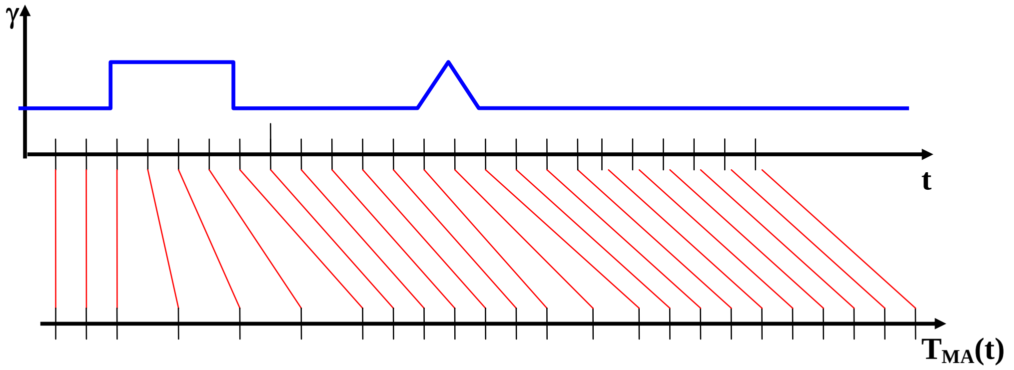

Figure 3.1 *Schematic explanation of the normalised intensity time. If a rate varies as a function of time, it can be compensated for by defining an intensity time that is dilated in areas of high rates, and compressed in times with low rates as a function of this intensity time, the resulting (normalised) rate will be constant, and it is possible to use Poisson statistics in the model.*

A challenge is the fact that the missile attack rate and the false alarm rate typically will vary independently. It is therefore important to have tight control with how the model is modified. We shall therefore review the model presented in chapter 3.1.

As before, we have two categories of cases:

- Category 1: There are flares left at time *t*
- Category 2: All flares have been used at time *t*

Cases of category 1 can occur if the number of missile attacks *i* is between *0* and *S-1*, and the number of false alarms *j* is between *0* and *S-1-i*. The total number of salvos fired then becomes $i + j \leq S$. The probability $P_{Survive,i}$ to survive such a case is as in the previous section given by:

$$P_{Survive,i} = \rho'^{\,i}_{n} \tag{3.20}$$

The probability of having flares left will now be given as:

$$p_{FL,i,j} = p_i\left(T_{MA}(t)\right) p_j\left(T_{FA}(t)\right), \text{ where } 0 \le i \le S-1, \text{ and } 0 \le j \le S-1-i, \tag{3.21}$$

We have here used the normalised probabilities:

$$p_j(T) = \frac{e^{-\gamma_0 T}\left(\gamma_0 T\right)^j}{j!} \tag{3.22}$$

This probability is, due to the normalisation, common for both false alarms and missile attacks (only the argument T changes). In cases of category 2, where all the flares are used by time t, we again define the time t_s where the last flare salvo was fired. As before, we have $0 < t_s \le t$. We now define the time interval during which the last flare salvo was fired as dt_s, and assumes this to be very short.

To survive the mission, it is necessary to survive both the attacks during the first time interval $0 \ldots t_s$ prior to the deployment of the last flare salvo, the time interval $t_s \ldots t_s + dt_s$ during which the last salvo was fired, and the remaining time interval $t_s + dt_s \ldots t$, during which there might also be any number of missile attacks. The survival probability of such a case is given as the product of the survival probability of all three event sequences.

Until time t_s, S-1 flare salvos have been fired, either because of false alarms, or because of missile attacks. The number of missile attacks i is then within the interval 0...S-1, and the number of false alarms j will be given as $j = S-1-i$.

The probability of surviving i missile attacks will as before be given as $P_{Survive,i} = \rho'^{\,i}_{n}$.

The probability of each of these cases to occur, meanwhile, will now be:

$$p_{FL,i,j=S-1-i}\left(t_s\right) = p_i\left(T_{MA}\left(t_s\right)\right) p_{S-1-i}\left(T_{FA}\left(t_s\right)\right), \text{ where } 0 \le i \le S-1, j = S-1-i \tag{3.23}$$

During the time interval where the last flare salvo is deployed, two things can happen:

- There is a missile attack, with a probability of occurrence $\gamma_{MA} dt_s$ and survival probability ρ'_r.
- There is a false alarm, with a probability of occurrence $\gamma_{FA} dt_s$ and survival probability 1.

It can be noted that r is used as an index instead of n, due to the possibility that the last salvo may not be complete. The probability of one of these cases to occur is given as:

$$P\left(\text{Probability of one MA or one FA during the interval } t_s \ldots t_s + dt_s\right) = dt_s\left(\gamma_{MA}\left(t_s\right) + \gamma_{FA}\left(t_s\right)\right) \tag{3.24}$$

During the interval after the deployment of the last flare salvo there might be any number of missile attacks. As there are no flares left, the survival probability for each missile attack is given as $\rho'_0 = \rho_0$. The probability for each of these cases to occur is now given by:

$$p\left(\text{Probability of k MA during the interval } t_s + dt_s \ldots t\right) = p_k\left(T_{MA}\left(t\right) - T_{MA}\left(t_s\right)\right), \text{k=0,1,....,}\infty \tag{3.25}$$

We must here remember to sum for all possible values of t_s.

The total mission survival probability P for both categories then becomes:

$$\begin{aligned} P = &\left(\sum_{i=0}^{S-1}\sum_{j=0}^{S-1-i} p_i\left(T_{MA}\left(t\right)\right)\rho'^{\,i}_n\, p_j\left(T_{FA}\left(t\right)\right)\right) \\ &+\int_0^t dt_s\left(\left(\gamma_{MA}\left(t_s\right)\rho'_r + \gamma_{FA}\left(t_s\right)\right)\sum_{i=0}^{S-1}\left(p_i\left(T_{MA}\left(t_s\right)\right)\rho'^{\,i}_n\, p_{S-1-i}\left(T_{FA}\left(t_s\right)\right)\right)\sum_{k=0}^{\infty} p_k\left(T_{MA}\left(t\right) - T_{MA}\left(t_s\right)\right)\rho_0^{\,k}\right) \end{aligned} \tag{3.26}$$

Inserting the normalised probability distributions, we get:

$$\begin{aligned} P = &\left(\sum_{i=0}^{S-1}\sum_{j=0}^{S-1-i} \frac{e^{-\gamma_0 T_{MA}(t)}\left(\gamma_0 T_{MA}\left(t\right)\right)^i}{i!}\rho'^{\,i}_n \frac{e^{-\gamma_0 T_{FA}(t)}\left(\gamma_0 T_{FA}\left(t\right)\right)^j}{j!}\right) \\ &+\int_0^t dt_s\left[\left(\gamma_{MA}\left(t_s\right)\rho'_r + \gamma_{FA}\left(t_s\right)\right)\sum_{i=0}^{S-1}\left(\frac{e^{-\gamma_0 T_{MA}(t_s)}\left(\gamma_0 T_{MA}\left(t_s\right)\right)^i}{i!}\rho'^{\,i}_n \frac{e^{-\gamma_0 T_{FA}(t_s)}\left(\gamma_0 T_{FA}\left(t_s\right)\right)^{S-1-i}}{\left(S-1-i\right)!}\right)\right. \\ &\left.\times\sum_{k=0}^{\infty}\frac{e^{-\gamma_0\left(T_{MA}(t) - T_{MA}(t_s)\right)}\left(\gamma_0\left(T_{MA}\left(t\right) - T_{MA}\left(t_s\right)\right)\right)^k}{k!}\rho_0^{\,k}\right] \end{aligned} \tag{3.27}$$

This expression can again be simplified. As long as we have not specified the definition of the normalised time $T_{MA}(t)$ and $T_{FA}(t)$, it is not possible to solve the integration analytically. Even if it is specified, it will in general not be possible to solve the integration analytically. It is, however, possible to solve the integration numerically if the expected missile attack and false alarm rates are known as a function of time. For completeness we now include the expression for the detection probability of the MAWS.

The step by step derivation of the general expression of the mission survival probability is given in appendix B. We finally obtain:

$$P = e^{-\gamma_0\left(T_{MA}(t)+T_{FA}(t)\right)}\left(\sum_{u=0}^{S-1}\frac{\left(\gamma_0 T_{FA}(t)+\rho'_n\,\gamma_0 T_{MA}(t)\right)^u}{u!}\right)$$
$$+\frac{\gamma_0^{\;S-1}e^{(\rho_0-1)\gamma_0 T_{MA}(t)}}{(S-1)!}\int_0^t dt_s\left[e^{-\gamma_0\left(T_{FA}(t_s)+\rho_0 T_{MA}(t_s)\right)}\left(\rho'_r\,\gamma_{MA}(t_s)+\gamma_{FA}(t_s)\right)\left(\rho'_n\,T_{MA}(t_s)+T_{FA}(t_s)\right)^{S-1}\right] \quad (3.28)$$
$$\rho'_n = \rho_0 + p_D\,p_{CME,n}\left(1-\rho_0\right)$$
$$\rho'_r = \rho_0 + p_D\,p_{CME,r}\left(1-\rho_0\right)$$

Alternatively, this can be expressed using an integral (see appendix B):

$$P = \left(e^{(\rho'_n-1)\gamma_0 T_{MA}(t)}\right.$$
$$+\frac{\gamma_0^{\;S-1}e^{-\gamma_0 T_{MA}(t)}}{(S-1)!}\int_0^t\left(dt_s\left[\rho'_n\,T_{MA}(t_s)+T_{FA}(t_s)\right]^{S-1}e^{-\gamma_0 T_{FA}(t_s)}\times\right.$$
$$\left.\left.\left[e^{\rho_0\gamma_0\left(T_{MA}(t)-T_{MA}(t_s)\right)}\left(\rho'_r\,\gamma_{MA}(t_s)+\gamma_{FA}(t_s)\right)-e^{\rho'_n\gamma_0\left(T_{MA}(t)-T_{MA}(t_s)\right)}\left(\rho'_n\,\gamma_{MA}(t_s)+\gamma_{FA}(t_s)\right)\right]\right]\right) \quad (3.29)$$

3.3 Multiple missile types and various missile firing conditions

Assuming it is possible to find the function of the expected variable missile attack rates, it is tempting to include the possibility of having a variation in the expected missile attack rates as a function of individual missile types. This would be interesting if several enemy groups, each group having their own weapons arsenal, occupy different areas to be overflown. At first glimpse, it may seem as a major task to refine the model by introducing different missile types. Looking more closely at the model, however, we find that we have already made a similar separation once, by introducing the competing false alarm rates. We now need some more terms to make the different expressions meaningful. We therefore define the following:

$P_{Survive,b,i}$: The probability of surviving *i* missile attacks from missile type *b*

$p_{b,j}(t)$: The probability for *j* different attacks from missile type *b* to take place within the time *t* (from t_0=0)

$\gamma_{MA,b}$: Expected average missile attack rate (number of attacks per unit time) from missile type *b*

$T_{MA,b}(t)$: Normalised intensity time for missile type *b*

$\rho_{b,n}$: Probability of surviving a missile attack from a missile type *b* when a *n*-flare salvo is deployed

By employing the same strategy as before, namely the separation of the sequences into two categories, we get:

- Category 1: There are flares left at time *t*
- Category 2: All flares have been used at time *t*

In category 1, the sum of the number of false alarms and different missile attacks must be equal to, or less than *S*-1.

In category 2, we separate the interval 0...t into three sub-intervals. In the first interval 0...t_s, the sum of missile attacks and false alarms must equal S-1. In the second interval t_s...t_s+dt_s, we must sum up the probabilities of each missile rate and false alarm rate and multiply each term with the corresponding missile attack survival probability. In the third interval t_s+dt_s...t, we must add together the probabilities of having k_b missile attacks from each of the missile types, each factor ranging from 0 to infinity, and each multiplied with their respective survival probabilities. The latter is greatly simplified by noting that the missile attacks are independent. We can thus take the product of the sums for each missile type.

The total mission survival probability can then be expressed as follows:

$$
\begin{aligned}
P = &\left(\sum_{k_1=0}^{S-1} \sum_{k_2=0}^{S-1-k_1} \cdots \sum_{k_b=0}^{S-1-\sum_{d=1}^{b-1} k_d} \cdots \sum_{k_B=0}^{S-1-\sum_{f=1}^{B-1} k_f} \sum_{j=0}^{S-1-\sum_{g=1}^{B} k_g} \left(\left(\prod_{h=1}^{B} p_{h,k_h}\left(T_{MA,h}(t)\right) \rho'_{h,n}{}^{k_h} \right) p_j\left(T_{FA}(t)\right) \right) \right. \\
&+ \int_0^t dt_s \left(\left[\sum_{m=1}^{B} \left(\gamma_{MA,m}(t_s) \rho'_{m,r} \right) + \gamma_{FA}(t_s) \right] \times \right. \\
&\quad \left[\sum_{k_1=0}^{S-1} \sum_{k_2=0}^{S-1-k_1} \cdots \sum_{k_b=0}^{S-1-\sum_{d=1}^{b-1} k_d} \cdots \sum_{k_B=0}^{S-1-\sum_{f=1}^{B-1} k_f} \left(\left(\prod_{h=1}^{B} p_{h,k_h}\left(T_{MA,h}(t_S)\right) \rho'_{h,n}{}^{k_h} \right) p_{S-1-\sum_{g=1}^{B} k_g} \left(T_{FA}(t_S)\right) \right) \right] \times \\
&\quad \left. \left. \left[\prod_{h=1}^{B} \sum_{k_h=0}^{\infty} p_{h,k_h}\left(T_{MA,h}(t) - T_{MA,h}(t_s)\right) \rho_{h,0}{}^{k_h} \right] \right) \right)
\end{aligned}
\tag{3.30}
$$

Inserting the Poisson distributions into this expression yields:

$$
\begin{aligned}
P = &\\
&\left(\sum_{k_1=0}^{S-1} \sum_{k_2=0}^{S-1-k_1} \cdot\cdot \sum_{k_b=0}^{S-1-\sum_{d=1}^{b-1} k_d} \cdot\cdot \sum_{k_B=0}^{S-1-\sum_{f=1}^{B-1} k_f} \sum_{j=0}^{S-1-\sum_{g=1}^{B} k_g} \left(\frac{e^{-\gamma_0 T_{FA}(t)} \left(\gamma_0 T_{FA}(t)\right)^j}{j!} \prod_{h=1}^{B} \frac{e^{-\gamma_0 T_{MA,h}(t)} \left(\gamma_0 T_{MA,h}(t)\right)^{k_h}}{k_h!} \rho'_{h,n}{}^{k_h} \right) \right. \\
&+ \int_0^t dt_s \left(\left[\gamma_{FA}(t_s) + \sum_{m=1}^{B} \left(\gamma_{MA,m}(t_s) \rho'_{m,r} \right) \right] \times \right. \\
&\left[\sum_{k_1=0}^{S-1} \sum_{k_2=0}^{S-1-k_1} \cdots\cdot \sum_{k_b=0}^{S-1-\sum_{d=1}^{b-1} k_d} \cdots \sum_{k_B=0}^{S-1-\sum_{f=1}^{B-1} k_f} \left(\frac{e^{-\gamma_0 T_{FA}(t_S)} \left(\gamma_0 T_{FA}(t_S)\right)^{S-1-\sum_{g=1}^{B} k_g}}{\left(S-1-\sum_{g=1}^{B} k_g \right)!} \prod_{h=1}^{B} \frac{e^{-\gamma_0 T_{MA,h}(t_S)} \left(\gamma_0 T_{MA,h}(t_S)\right)^{k_h}}{k_h!} \rho'_{h,n}{}^{k_h} \right) \right] \times \\
&\left. \left. \left[\prod_{h=1}^{B} \sum_{k_h=0}^{\infty} \frac{e^{-\gamma_0 \left(T_{MA,h}(t) - T_{MA,h}(t_s)\right)} \left(\gamma_0 \left(T_{MA,h}(t) - T_{MA,h}(t_s)\right)\right)^{k_h}}{k_h!} \rho_{h,0}{}^{k_h} \right] \right) \right)
\end{aligned}
\tag{3.31}
$$

This expression, however impressing, can be attacked in much the same way as was the case for expression (3.26). The derivation of the resulting expression can be found in appendix C. We finally get:

$$P = e^{-\gamma_0\left(T_{FA}(t)+\sum_{h=1}^{B}T_{MA,h}(t)\right)}\left(\sum_{K=0}^{S-1}\frac{\left(\gamma_0 T_{FA}(t)+\sum_{h=1}^{B}\rho'_{h,n}\,\gamma_0 T_{MA,h}(t)\right)^K}{K!}\right)+\frac{\gamma_0{}^{S-1}e^{\gamma_0\sum_{h=1}^{B}(\rho_{h,0}-1)T_{MA,h}(t)}}{(S-1)!}\times$$

$$\int_0^t dt_s\left(\left(\gamma_{FA}(t_s)+\sum_{m=1}^{B}\left(\gamma_{MA,m}(t_s)\,\rho'_{m,r}\right)\right)e^{-\gamma_0\left(T_{FA}(t_S)+\sum_{h=1}^{B}\rho_{h,0}T_{MA,h}(t_s)\right)}\left(T_{FA}(t_S)+\sum_{g=1}^{B}\left(\rho'_{g,n}\,T_{MA,g}(t_S)\right)\right)^{S-1}\right) \quad (3.32)$$

$$\rho'_{b,n} = \rho_{b,0} + p_{D,b}\,p_{CME,b,n}\left(1-\rho_{b,0}\right)$$
$$\rho'_{r} = \rho_{b,0} + p_{D,b}\,p_{CME,b,r}\left(1-\rho_{b,0}\right)$$

This model, taking into account several missile types, could be extended to incorporate several missile types in different attack configurations. As countermeasures performance against various missiles can be aspect dependent or dependent on distance, each case could be treated as if they were different missile types. Each expected missile attack rate could then be calculated as a function of time in a model taking into account geographical information.

3.4 Optimisation of flare programs

It may be that a constant number of flares in the program will be the best *à priori* choice for optimising the mission survival probability if the expected missile attack rates and the false alarm rate are constant, although this has not been proven here. This will generally **not**, however, be the case if the false alarm rate and the various expected missile attack rates are independent functions of time. This can most easily be seen in the case where at times the missile attack rate is zero (or extremely low), whereas in other places it is high. In this case, there is little point in squandering flares in regions where the only deployment will be due to false alarms, whereas at the same time the use of large flare programs in high risk areas during the mission would seem reasonable. This means that optimizing the flare program should at any time be done for all times. This could be done by dividing the mission time interval into smaller intervals and testing the total mission survival probability for each change in time interval. Ideally, this optimisation should be done dynamically. It is thus seen that this kind of optimisation could make significant demands on the DAS in terms of computing power.

4 PARAMETER SENSITIVITY OF THE MISSION SURVIVAL PROBABILITY

We will now start with the model developed in chapter 3.1, (with constant expected missile attack rates and false alarm rates and a complete last salvo) and examine the degree to which input parameters influence the mission survival probability. We will here limit the study to the continuous input parameters t, γ_{MA}, γ_{FA}, p_D, ρ_0 and $p_{CME,n}$. One way to study the parameter sensitivity is by looking at the partial derivatives of P with respect to the different parameters. Another is to look at functions that are reasonably good approximations to P. For the evaluation of the partial derivatives as well as finding approximate functions, it can be beneficial to start with the two alternative forms of the expression, as given by equations (3.13) and (3.14):

$$\textit{Summation form}:$$
$$P=\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^S e^{(\rho_0-1)\gamma_{MA}t}+\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^S e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{u-S}-\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S}\right)\right)$$

$$\textit{Integral form}:$$
$$P=e^{(\rho'_n-1)\gamma_{MA}t}+\frac{\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^S}{(S-1)!}\int_0^t\left(dt_s t_s^{S-1}\left(e^{(\rho_0-1)\gamma_{MA}t-(\rho_0\gamma_{MA}+\gamma_{FA})t_s}-e^{(\rho'_n-1)\gamma_{MA}t-(\rho'_n\gamma_{MA}+\gamma_{FA})t_s}\right)\right)$$
$$\rho'_n=\rho_0+p_D p_{CME,n}\left(1-\rho_0\right)$$

Mathematically, obtaining the partial derivatives is a fairly straightforward task, and they will therefore just be written out directly, apart from one exception.

4.1 Sensitivity to the mission duration *t*

We start by looking at the limiting values of P as a function of time. The simplest expression is found for small values of t. The probability of empty dispensers is then very small, and we get:

$$\lim_{t\to 0}P=\lim_{t\to 0}e^{(\rho'_n-1)\gamma_{MA}t}=1 \tag{4.1}$$

We have here included the dominating terms. This expression can be found by using both expressions for P, even though the use of the integral expression is significantly simpler than using the sum version of P.

For large values of t, P tends towards 0. For large values of t we have:

$$\lim_{t\to\infty}P=\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^S\lim_{t\to\infty}e^{(\rho_0-1)\gamma_{MA}t}=0 \tag{4.2}$$

This limit is most easily found using the sum version of P.

We now compare the functions corresponding to the dominating terms found at the extreme limits of *t* with the mission survival probability *P*. Starting with the first function G_1 defined by:

$$G_1(t) = e^{(\rho'_n - 1)\gamma_{MA} t} \tag{4.3}$$

we find in appendix D that:

$$e^{(\rho'_n - 1)\gamma_{MA} t} - P \geq 0 \tag{4.4}$$

for all *t*. The expression $e^{(\rho'_n - 1)\gamma_{MA} t}$ is therefore an upper limit for the mission survival probability *P*. This sounds intuitively reasonable, as the function corresponds to a mission survival probability with an infinite flare capacity.

We now look at:

$$G_2(t) = \left(\frac{\rho'_n \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}} \right)^S e^{(\rho_0 - 1)\gamma_{MA} t} \tag{4.5}$$

In appendix D it is also found that:

$$\left(\frac{\rho'_n \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}} \right)^S e^{(\rho_0 - 1)\gamma_{MA} t} - P \geq 0 \tag{4.6}$$

for all values of *t*.

This function corresponds to the survival probability without the use of any countermeasures, but with a modifying factor compensating for the firing of the first S salvos that provides an increased survival probability.

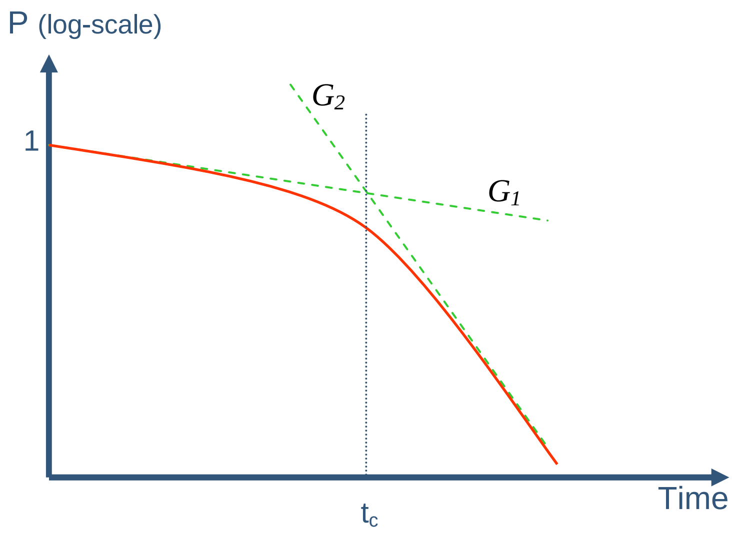

Figure 4.1 *Schematic time development of the mission survival probability P (red curve). The two green curves indicates the two upper limiting functions $G_1(t)$ and $G_2(t)$ described in the text. Note the logarithmic scale of the y-axis, and hence that the x-axis does not cross the y-axis at y=0.*

A plot of the logarithm of the mission survival probability as a function of (linear) time will appear below the lines generated by these two limiting functions. To find how well the two limiting curves mimick the probability curve, it is interesting to examine what happens below the point where the curves of the two limiting functions cross each other. This is shown in Figure 4.1.

The crossing point is found by inserting for $G_1(t) = G_2(t)$:

$$e^{(\rho'_n - 1)\gamma_{MA} t_c} = \left(\frac{\rho'_n \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}} \right)^S e^{(\rho_0 - 1)\gamma_{MA} t_c} \Leftrightarrow t_c = \frac{S}{(\rho'_n - \rho_0)\gamma_{MA}} \ln \left(\frac{\rho'_n \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}} \right) \tag{4.7}$$

The values of the limiting functions are then:

$$G_1(t_c) = G_2(t_c) = e^{\frac{(\rho'_n - 1)S}{(\rho'_n - \rho_0)} \ln \left(\frac{\rho'_n \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}} \right)} = \left(\frac{\rho'_n \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}} \right)^{\frac{(\rho'_n - 1)}{(\rho'_n - \rho_0)} S} \tag{4.8}$$

Inserting for t in the two (equivalent) expressions for P in the equations (3.13) and (3.14) , we find that:

$$P(t_c)=\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^{\frac{(\rho'_n-1)}{(\rho'_n-\rho_0)}S}$$
$$+\sum_{u=0}^{S-1}\left(\frac{\left(\frac{S}{(\rho'_n-\rho_0)\gamma_{MA}}\ln\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)\right)^u}{u!}\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^{\frac{-(\gamma_{MA}+\gamma_{FA})}{(\rho'_n-\rho_0)\gamma_{MA}}S}\right. \tag{4.9}$$
$$\left.\times\left((\rho'_n\gamma_{MA}+\gamma_{FA})^u-\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^S(\rho_0\gamma_{MA}+\gamma_{FA})^u\right)\right)$$

$$P(t_c)=\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^{\frac{(\rho'_n-1)}{(\rho'_n-\rho_0)}S}$$
$$+\frac{(\rho'_n\gamma_{MA}+\gamma_{FA})^S}{(S-1)!}\int_0^{\frac{S}{(\rho'_n-\rho_0)\gamma_{MA}}\ln\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)}\left(dt_s t_s^{S-1}\left(e^{(\rho_0-1)\gamma_{MA}t-(\rho_0\gamma_{MA}+\gamma_{FA})t_s}-e^{(\rho'_n-1)\gamma_{MA}t-(\rho'_n\gamma_{MA}+\gamma_{FA})t_s}\right)\right) \tag{4.10}$$

In cases where $\gamma_{FA}\gg S\gamma_{MA}$, we can use the approximations $\ln(1+x)\approx x$ and $e^x\approx 1+x$ for small values of x. We then obtain:

$$t_c\approx\frac{S}{\gamma_{FA}} \tag{4.11}$$

This value of t can be thought of as a critical time, as it corresponds to the expected time at which all the countermeasures dispensers have been emptied. Inserting, this value, we get:

$$G_1(t_c)=G_2(t_c)\approx 1+(\rho'_n-1)S\frac{\gamma_{MA}}{\gamma_{FA}} \tag{4.12}$$

$$P(t_c)\approx\left(1+S\frac{\gamma_{MA}}{\gamma_{FA}}(\rho'_n-1)\right)\left(1-S\frac{\gamma_{MA}}{\gamma_{FA}}(\rho'_n-\rho_0)\left(\frac{e^{-S}S^S}{S!}\right)\right)$$
$$\approx\left(1+S\frac{\gamma_{MA}}{\gamma_{FA}}\left(\rho'_n-1-(\rho'_n-\rho_0)\left(\frac{e^{-S}S^S}{S!}\right)\right)\right) \tag{4.13}$$

The ratio P/G at this point can be of interest. We have:

$$\frac{P(t_c)}{G(t_c)}\approx\left(1-S\frac{\gamma_{MA}}{\gamma_{FA}}(\rho'_n-\rho_0)\left(\frac{e^{-S}S^S}{S!}\right)\right)\approx 1-0.4\frac{\gamma_{MA}}{\gamma_{FA}}(\rho'_n-\rho_0)\sqrt{S} \tag{4.14}$$

where we have used Stirling’s formula for (S!). The expression:

$$\left(\frac{e^{-S}S^S}{S!}\right) \approx \frac{1}{\sqrt{2\pi S}} \approx \frac{0.4}{\sqrt{S}} \tag{4.15}$$

has accuracy better than 10% for $S \geq 1$, better than 5% for $S \geq 2$ and better than 2.5% for $S \geq 4$. We now look at the time derivative of the mission survival probability to look at changes as a function of time. Because we could imagine changing other parameter values at the same time, we choose to look at the partial derivative. Mathematically, this can be written in two ways, depending on which expression is used for the mission survival probability (the integral form or the summation form). Using the summation form we have:

$$\begin{aligned}\frac{\partial P}{\partial t} = &\\ &\gamma_{MA}\left(\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^S(\rho_0-1)e^{(\rho_0-1)\gamma_{MA}t} + \frac{(\rho'_n-\rho_0)(\gamma_{MA}+\gamma_{FA})}{(\rho_0\gamma_{MA}+\gamma_{FA})(S-1)!}(\rho'_n\gamma_{MA}t+\gamma_{FA}t)^{S-1}e^{-(\gamma_{MA}+\gamma_{FA})t}\right.\\ &\left.+(\rho'_n\gamma_{MA}+\gamma_{FA})^S e^{-(\gamma_{MA}+\gamma_{FA})t}\sum_{u=0}^{S-2}\frac{t^u}{u!}\left((\rho'_n-1)(\rho'_n\gamma_{MA}+\gamma_{FA})^{u-S}-(\rho_0-1)(\rho_0\gamma_{MA}+\gamma_{FA})^{u-S}\right)\right)\end{aligned} \tag{4.16}$$

Alternatively we can derive it from the integral form of P. We then get:

$$\begin{aligned}\frac{\partial P}{\partial t} = &(\rho'_n-1)\gamma_{MA}e^{(\rho'_n-1)\gamma_{MA}t}\\ &+\frac{(\rho'_n\gamma_{MA}+\gamma_{FA})^S}{(S-1)!}\left(\gamma_{MA}e^{-\gamma_{MA}t}\int_0^t\left(dt_s t_s^{S-1}e^{-\gamma_{FA}t_s}\left[(\rho_0-1)\left(e^{\rho_0\gamma_{MA}(t-t_s)}\right)-(\rho'_n-1)\left(e^{\rho'_n\gamma_{MA}(t-t_s)}\right)\right]\right)\right)\end{aligned} \tag{4.17}$$

For small values of t this time derivative is can be approximated by:

$$\lim_{t\to 0}\frac{\partial P}{\partial t} = (\rho'_n-1)\gamma_{MA} \tag{4.18}$$

an expression of the (negative) rate of effective missile attacks when using countermeasures. For large values of t we have:

$$\lim_{t\to 0}\frac{\partial P}{\partial t} = \lim_{t\to 0}\left((\rho_0-1)\gamma_{MA}e^{(\rho_0-1)\gamma_{MA}t}\left(\frac{\gamma_{FA}+\rho'_n\gamma_{MA}}{\gamma_{FA}+\rho_0\gamma_{MA}}\right)^S\right) = 0 \tag{4.19}$$

an expression agreeing with the limiting value obtained for P for large values of t.

If the effective missile attack survival probability using a small flare program is higher than that for a larger program it can be showed that the small program gives the highest mission survival probability. If the largest flare program leads to the highest effective missile attack survival probability, the picture is somewhat more complicated, as will be shown in the following.

4.1.1 The case $\rho'_n > \rho'_m$, $n > m$, $S_n < S_m$

For very small values of *t* (short missions), a high effective missile attack survival probability always gives the better mission survival probability. For larger values of *t*, though, there might be differences, according to parameters. We will now compare the mission survival of two flare programs obeying the conditions indicated in the chapter title. The difference between the two programs is given by:

$$\begin{aligned} P_n - P_m = & \left(\frac{\rho'_n \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}}\right)^{S_n} e^{(\rho_0 - 1)\gamma_{MA} t} - \left(\frac{\rho'_m \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}}\right)^{S_m} e^{(\rho_0 - 1)\gamma_{MA} t} \\ & + \left(\rho'_n \gamma_{MA} + \gamma_{FA}\right)^{S_n} e^{-(\gamma_{MA}+\gamma_{FA})t} \left(\sum_{u=0}^{S_n - 1} \frac{t^u}{u!}\left(\left(\rho'_n \gamma_{MA} + \gamma_{FA}\right)^{u-S_n} - \left(\rho_0 \gamma_{MA} + \gamma_{FA}\right)^{u-S_n}\right)\right) \\ & - \left(\rho'_m \gamma_{MA} + \gamma_{FA}\right)^{S_m} e^{-(\gamma_{MA}+\gamma_{FA})t} \left(\sum_{u=0}^{S_m - 1} \frac{t^u}{u!}\left(\left(\rho'_m \gamma_{MA} + \gamma_{FA}\right)^{u-S_m} - \left(\rho_0 \gamma_{MA} + \gamma_{FA}\right)^{u-S_m}\right)\right) \end{aligned} \tag{4.20}$$

If:

$$\left(\frac{\gamma_{FA} + \rho'_n \gamma_{MA}}{\gamma_{FA} + \rho_0 \gamma_{MA}}\right)^{S_n} \geq \left(\frac{\gamma_{FA} + \rho'_m \gamma_{MA}}{\gamma_{FA} + \rho_0 \gamma_{MA}}\right)^{S_m} \tag{4.21}$$

we can rearrange the terms to obtain:

$$\begin{aligned} & P_n - P_m = \\ & \underbrace{\left(\left(\left(\frac{\rho'_n \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}}\right)^{S_n} - \left(\frac{\rho'_m \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}}\right)^{S_m}\right) e^{(\rho_0 - 1)\gamma_{MA} t} \left(1 - \left(\sum_{u=0}^{S_n - 1} \frac{e^{-(\rho_0 \gamma_{MA} + \gamma_{FA})t}\left(\left(\rho_0 \gamma_{MA} + \gamma_{FA}\right)t\right)^u}{u!}\right)\right)\right)}_{\geq 0} \\ & + e^{-(\gamma_{MA}+\gamma_{FA})t} \underbrace{\left(\sum_{u=0}^{S_n - 1} \frac{t^u}{u!}\left(\left(\rho'_n \gamma_{MA} + \gamma_{FA}\right)^u - \left(\rho'_m \gamma_{MA} + \gamma_{FA}\right)^u\right)\right)}_{>0} \\ & + e^{-(\gamma_{MA}+\gamma_{FA})t} \left(\sum_{u=S_n}^{S_m - 1} \frac{t^u}{u!} \underbrace{\left(\left(\frac{\rho'_m \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}}\right)^{S_m} \left(\rho_0 \gamma_{MA} + \gamma_{FA}\right)^u - \left(\rho'_m \gamma_{MA} + \gamma_{FA}\right)^u\right)}_{>0}\right) \\ & > 0 \end{aligned} \tag{4.22}$$

In the opposite case, if:

$$\left(\frac{\gamma_{FA} + \rho'_n \gamma_{MA}}{\gamma_{FA} + \rho_0 \gamma_{MA}}\right)^{S_n} < \left(\frac{\gamma_{FA} + \rho'_m \gamma_{MA}}{\gamma_{FA} + \rho_0 \gamma_{MA}}\right)^{S_m} \tag{4.23}$$

we get:

$$P_n - P_m =$$

$$\underbrace{\left(\left(\frac{\rho'_n \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}}\right)^{S_n} - \left(\frac{\rho'_m \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}}\right)^{S_m}\right) e^{(\rho_0 - 1)\gamma_{MA} t} \left(1 - \left(\sum_{u=0}^{S_n - 1} \frac{e^{-(\rho_0 \gamma_{MA} + \gamma_{FA})t} \left(\left(\rho_0 \gamma_{MA} + \gamma_{FA}\right) t\right)^u}{u!}\right)\right)}_{<0}$$

$$+ e^{-(\gamma_{MA} + \gamma_{FA})t} \underbrace{\left(\sum_{u=0}^{S_n - 1} \frac{t^u}{u!} \left(\left(\rho'_n \gamma_{MA} + \gamma_{FA}\right)^u - \left(\rho'_m \gamma_{MA} + \gamma_{FA}\right)^u\right)\right)}_{>0} \qquad (4.24)$$

$$+ e^{-(\gamma_{MA} + \gamma_{FA})t} \left(\sum_{u=S_n}^{S_m - 1} \frac{t^u}{u!} \underbrace{\left(\left(\frac{\rho'_m \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}}\right)^{S_m} \left(\rho_0 \gamma_{MA} + \gamma_{FA}\right)^u - \left(\rho'_m \gamma_{MA} + \gamma_{FA}\right)^u\right)}_{>0}\right)$$

For small values of *t*, we get:

$$\lim_{t \to 0^+} \left(P_n - P_m\right) = \lim_{t \to 0^+} \left(t \gamma_{MA} \left(\rho'_n - \rho'_m\right)\right) > 0 \qquad (4.25)$$

Conversely, for large values of *t*, we get:

$$\lim_{t \to \infty} \left(P_n - P_m\right) = \lim_{t \to \infty} \left(\left(\left(\frac{\rho'_n \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}}\right)^{S_n} - \left(\frac{\rho'_m \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}}\right)^{S_m}\right) e^{(\rho_0 - 1)\gamma_{MA} t}\right) < 0 \qquad (4.26)$$

There will therefore be at least one crossing of the two curves for the mission survival probability if:

$$\left(\frac{\gamma_{FA} + \rho'_n \gamma_{MA}}{\gamma_{FA} + \rho_0 \gamma_{MA}}\right)^{S_n} < \left(\frac{\gamma_{FA} + \rho'_m \gamma_{MA}}{\gamma_{FA} + \rho_0 \gamma_{MA}}\right)^{S_m} \qquad (4.27)$$

The crossing of the curves does not have to occur for practical values of *t*, though. *The model nevertheless shows that it is possible to get into situations where it is desirable to make a choice of the flare program setting as a function of mission duration. This also means that a dynamic setting may also be required to get an optimum performance.*

4.1.2 Summary

The expression of the mission survival probability *P* is relatively complicated. It is nevertheless possible to find upper limits $G_1(t)$ and $G_2(t)$ for *P* which are much simpler, and which have the same value and derivative in the extreme values of *t*. We have the following equations:

- $P(t) \le G_1(t) = e^{(\rho'_n - 1)\gamma_{MA} t}$
- $P(t) \le G_2(t) = \left(\dfrac{\rho'_n \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}} \right)^S e^{(\rho_0 - 1)\gamma_{MA} t}$
- $P(0) = G_1(0)$
- $P(t) \to G_2(t), t \to \infty$

For $\dfrac{S\gamma_{MA}}{\gamma_{FA}} \ll 1$:

- $t_c \approx \dfrac{S}{\gamma_{FA}}$
- $G_1(t_c) = G_2(t_c) \approx 1 - (1 - \rho'_n) \dfrac{S\gamma_{MA}}{\gamma_{FA}}$
- $\dfrac{P(t_c)}{G(t_c)} \approx 1 - 0.4 \dfrac{\gamma_{MA}}{\gamma_{FA}} (\rho'_n - \rho_0) \sqrt{S}$

These relatively easy expressions related to the mission survival probability *P* are well suited to find ”back of the envelope” solutions and are also well suited for graphical presentation, to give a feeling with what kind of performance to expect, for instance in educational use.

To compare the performance of two different countermeasures programs, we start by defining two sequences:

Sequence name	# of flares per salvo	# of salvos available	Effective missile attack survival probability
SEQ-n	*n*	S_n	ρ'_n
SEQ-m	*m*	S_m	ρ'_m

We can now create a table to sum up which countermeasures sequence to choose, given the program properties. It should be noted that the chart is symmetric, but all possible values have been included for completeness. The usual case when comparing flare programs with different flare counts would be that the highest flare count would lead to the highest effective missile attack survival probability.

Criteria			Best sequence
	$\rho'_n \leq \rho'_m$		*SEQ-m*
$n > m$	$\rho'_n > \rho'_m$	$\left(\frac{\gamma_{FA}+\rho'_n\gamma_{MA}}{\gamma_{FA}+\rho_0\gamma_{MA}}\right)^{S_n} \geq \left(\frac{\gamma_{FA}+\rho'_m\gamma_{MA}}{\gamma_{FA}+\rho_0\gamma_{MA}}\right)^{S_m}$	*SEQ-n*
		$\left(\frac{\gamma_{FA}+\rho'_n\gamma_{MA}}{\gamma_{FA}+\rho_0\gamma_{MA}}\right)^{S_n} < \left(\frac{\gamma_{FA}+\rho'_m\gamma_{MA}}{\gamma_{FA}+\rho_0\gamma_{MA}}\right)^{S_m}$	Depending on *t*
	$\rho'_n < \rho'_m$		*SEQ-m*
$n = m$	$\rho'_n = \rho'_m$		Equal performance
	$\rho'_n > \rho'_m$		*SEQ-n*
$n < m$	$\rho'_n < \rho'_m$	$\left(\frac{\gamma_{FA}+\rho'_n\gamma_{MA}}{\gamma_{FA}+\rho_0\gamma_{MA}}\right)^{S_n} > \left(\frac{\gamma_{FA}+\rho'_m\gamma_{MA}}{\gamma_{FA}+\rho_0\gamma_{MA}}\right)^{S_m}$	Depending on *t*
		$\left(\frac{\gamma_{FA}+\rho'_n\gamma_{MA}}{\gamma_{FA}+\rho_0\gamma_{MA}}\right)^{S_n} \leq \left(\frac{\gamma_{FA}+\rho'_m\gamma_{MA}}{\gamma_{FA}+\rho_0\gamma_{MA}}\right)^{S_m}$	*SEQ-m*
	$\rho'_n \geq \rho'_m$		*SEQ-n*

4.2 Sensitivity to the missile attack rate γ_{MA}

It is possible to do something with the missile attack probability per unit time through mission planning. It is therefore interesting to examine the effect of this rate on the mission survival probability. It could for instance be possible to make quantitative requirements for the mission survival probability when planning a mission, and plan the mission accordingly such that the parameters given are within acceptable limits.

Intuitively, an increase in missile attack rate leads to a reduced mission survival probability, and hence the partial derivative value must always be negative. Normally, there is a significant uncertainty linked to the attack rate, both because it is typically very low, and because its estimate is based on many uncertain factors.

Mathematically, the partial derivative is given as:

$$\begin{aligned}\frac{\partial P}{\partial \gamma_{MA}} &= (\rho'_n - 1)\, t e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{\left((\gamma_{FA}+\rho'_n\gamma_{MA})t\right)^u}{u!}\right) - e^{-(\gamma_{MA}+\gamma_{FA})t}\, t\rho'_n \frac{(\gamma_{FA}+\rho'_n\gamma_{MA})^{S-1}(t)^{S-1}}{(S-1)!} \\ &+ \left(S\rho'_n + t(\rho_0-1)(\gamma_{FA}+\rho'_n\gamma_{MA})\right)\frac{(\gamma_{FA}+\rho'_n\gamma_{MA})^{S-1} e^{(\rho_0-1)\gamma_{MA}t}}{(S-1)!}\int_0^t dt_s\, t_s^{S-1} e^{-(\rho_0\gamma_{MA}+\gamma_{FA})t_s} \\ &- \frac{\rho_0(\gamma_{FA}+\rho'_n\gamma_{MA})^S e^{(\rho_0-1)\gamma_{MA}t}}{(S-1)!}\left(\int_0^t dt_s\, t_s^{S} e^{-(\rho_0\gamma_{MA}+\gamma_{FA})t_s}\right)\end{aligned} \quad (4.28)$$

The uncertainty is nevertheless likely to be of such a big importance that the variations in this parameter cannot be considered to be a mere perturbation, limiting the use of the partial derivative.

The simplified expressions for *P(t)* summarised in chapter 4.1.2 shows that the changes in mission survival probability as a function of small changes in γ_{MA} is proportional to γ_{MA}, a result which intuitively seems reasonable.

We now focus on two parameters that could be adjusted either prior to a mission or throughout the mission. A missile approach warning system (MAWS) is generally not perfect, and will be a limiting factor in the overall performance of the self protection system, particularly if the flare decoy efficiency is high. Due to fundamental noise in the system, a compromise between a high detection probability and a low false alarm rate will have to be made for the MAWS.

4.3 Sensitivity to variations in the false alarm rate γ_{FA} and the detection probability p_D

The partial derivative of the mission survival probability with respect to the false alarm rate γ_{FA} is given as:

$$\frac{\partial P}{\partial \gamma_{FA}} = e^{(\rho_0 - 1)\gamma_{MA} t} \frac{S\gamma\left(\rho_0 - \rho'_n\right)}{\left(\rho_0\gamma_{MA} + \gamma_{FA}\right)^2}\left(\frac{\rho'_n \gamma_{MA} + \gamma_{FA}}{\rho_0\gamma_{MA} + \gamma_{FA}}\right)^{S-1}\left(1 - \sum_{u=0}^{S} \frac{e^{-(\rho_0\gamma_{MA} + \gamma_{FA})t}\left(\left(\rho_0\gamma_{MA} + \gamma_{FA}\right)t\right)^u}{u!}\right) \tag{4.29}$$

The derivation of this is shown in appendix E.

The partial derivative of the mission survival probability with respect to the missile detection probability p_D is given as:

$$\begin{aligned}\frac{\partial P}{\partial p_D} &= p_{CME,n}\left(1 - \rho_0\right) \times \\ &\left(\frac{S\gamma_{MA} e^{(\rho_0 - 1)\gamma_{MA} t}}{\rho_0\gamma_{MA} + \gamma_{FA}}\left(\frac{\left(\rho_0 + p_D p_{CME,n}\left(1 - \rho_0\right)\right)\gamma_{MA} + \gamma_{FA}}{\rho_0\gamma_{MA} + \gamma_{FA}}\right)^{S-1}\left(1 - \sum_{u=0}^{S-1} \frac{e^{-(\rho_0\gamma_{MA} + \gamma_{FA})t}\left(\left(\rho_0\gamma_{MA} + \gamma_{FA}\right)t\right)^u}{u!}\right)\right. \\ &\left. + t\gamma_{MA} e^{-(\gamma_{MA} + \gamma_{FA})t} \sum_{u=0}^{S-2} \frac{\left(\left(\left(\rho_0 + p_D p_{CME,n}\left(1 - \rho_0\right)\right)\gamma_{MA} + \gamma_{FA}\right)t\right)^u}{u!}\right)\end{aligned} \tag{4.30}$$

The expression itself may not provide much insight, but is important, as it is possible to use it together with the partial derivative with respect to the false alarm rate to optimise the performance of the MAWS. An incoming missile is detected when the signal produced by the incoming missile exceeds a threshold value. The lower the threshold, the higher is the probability that the incoming missile is detected. On the other hand, lowering the threshold closes the gap between the noise floor and the threshold, leading to an increase in the false alarm rate. Therefore, the detection probability is normally a monotonically increasing function of the false alarm rate, and an optimum setting of the noise threshold can be found by solving the equation:

$$\frac{\partial P}{\partial p_D}\frac{\partial p_D}{\partial \gamma_{FA}}+\frac{\partial P}{\partial \gamma_{FA}}=0 \qquad (4.31)$$

This equation will typically have two solutions, of which one, found at $\gamma_{FA}=\infty$ is uninteresting, while the other is where the parameter values give optimum performance.

5 UNCERTAINTY IN THE VALUE OF THE MISSION SURVIVAL PROBABILITY

In order to make practical use of the model previously presented, it is important to be aware of the uncertainty that follows the model. These uncertainties can be split into two main categories:

1. **Uncertainty due to parameter estimates.** Typically, rates with low values are hard to estimate. Furthermore, rates which are difficult to measure directly may also result in significant uncertainty. The main example here is the missile attack rate, which typically is extremely low in most scenarios, and which is also difficult to measure. First, missile attacks are often not detected, and secondly, it is not easy to distinguish one type of weapon with another in the heat of the fight. Another example is the missile attack survival probability, given that no countermeasures are deployed. First, this is not the kind of test one would actively seek to measure; secondly, such performance will heavily rely almost as much upon the missile operator skill as of the missile quality. Another issue related to the establishment of such average numbers is that the mission survival probability is not a linear function of the parameter values. To account for this in a rigorous way, it is necessary to establish a probability distribution for the different possible values and integrate the mission survival probability over all possible parameter values. Having taken all these precautions, the uncertainty may still be fundamentally due to the fact that there is no established method available to estimate the input values, or expected probability distributions for these.
2. **Statistical uncertainty.** The results of stochastic processes such as these have built-in variations by nature. Although the uncertainty of average values obtained may become small, parameters with small values such as the probability of missile attack per unit time will typically result in output that remains cluttered with spurious effects and coincidences, even after major military campaigns, making extraction of meaningful results difficult.

What we until now has called the mission survival probability P really is the expectance value $\langle P_{Survive}\rangle$ of the survival probability $P_{Survive}$ to survive a given sequence of events during a mission, averaged over all possible sequences of events. To improve the understanding of the mission survival as a statistical process we now establish the higher moments of the probability distribution. These moments are given as $\langle P_{Survive}{}^{m}\rangle$. They are important, because all the statistical properties of the mission survival probability as a probability distribution are described by the statistical moments. The most common statistical property used is (apart from the average value) the variance σ^2, from which the standard deviation σ is derived. The variance is defined as $\langle(P_{Survive}-\langle P_{Survive}\rangle)^2\rangle$, averaged over all possible sequences of events. The linearity properties mean that we can express the variance as:

$$\sigma^2 = \left\langle \left(P_{Survive} - \left\langle P_{Survive} \right\rangle\right)^2 \right\rangle = \left\langle P_{Survive}{}^2 - 2P_{Survive}\left\langle P_{Survive}\right\rangle + \left\langle P_{Survive}\right\rangle^2 \right\rangle = \left\langle P_{Survive}{}^2 \right\rangle - \left\langle P_{Survive}\right\rangle^2 \quad (5.1)$$

We now look at the three models developed in chapters 2, 3.1 and 3.2, starting with the basic model.

5.1 Higher order statistical moments

A departure point for expressing the m^{th} statistical moment for the basic model developed in chapter 2 is based on a modification of the expression in equation (2.6). We have:

$$\begin{aligned}
\left\langle P_{survive}{}^m \right\rangle &= \sum_{i=0}^{\infty} \frac{e^{-\gamma_{MA}t}\left(\gamma_{MA}t\right)^i}{i!} P_{survive,i}{}^m \\
&= \sum_{i=0}^{S} \frac{e^{-\gamma_{MA}t}\left(\gamma_{MA}t\right)^i}{i!} \rho_n{}^{mi} + \sum_{i=S+1}^{\infty} \frac{e^{-\gamma_{MA}t}\left(\gamma_{MA}t\right)^i}{i!} \left(\frac{\rho_n}{\rho_0}\right)^{mS} \rho_0{}^{mi} \\
&= \sum_{i=0}^{S} \frac{e^{-\gamma_{MA}t}\left(\gamma_{MA}t\right)^i}{i!} \left(\rho_n{}^{mi} - \left(\frac{\rho_n}{\rho_0}\right)^{mS} \rho_0{}^{mi}\right) + \sum_{i=0}^{\infty} \frac{e^{-\gamma_{MA}t}\left(\gamma_{MA}t\right)^i}{i!} \left(\frac{\rho_n}{\rho_0}\right)^{mS} \rho_0{}^{mi} \\
&= e^{-\gamma t}\left(\sum_{i=0}^{S} \left(\frac{\left(\rho_n{}^m \gamma_{MA} t\right)^i - \left(\frac{\rho_n}{\rho_0}\right)^{mS} \left(\rho_0{}^m \gamma_{MA} t\right)^i}{i!} \right) + \left(\frac{\rho_n}{\rho_0}\right)^{mS} \sum_{i=0}^{\infty} \frac{\left(\rho_0{}^m \gamma_{MA} t\right)^i}{i!} \right)
\end{aligned} \quad (5.2)$$

Simplifying the infinite sum, we end up with the following expression:

$$\left\langle P_{survive}{}^m \right\rangle = \left(\frac{\rho_n}{\rho_0}\right)^{mS} e^{\left(\rho_0{}^m - 1\right)\gamma_{MA}t} + e^{-\gamma_{MA}t} \sum_{i=0}^{S} \left(\frac{\left(\rho_n{}^m \gamma_{MA} t\right)^i - \left(\frac{\rho_n}{\rho_0}\right)^{mS} \left(\rho_0{}^m \gamma_{MA} t\right)^i}{i!} \right) \quad (5.3)$$

Comparing this result with the expression obtained in chapter 2, we find that the expression (2.7) is modified through the following transformation:

$$\begin{aligned} \rho_n &\to \rho_n{}^m \\ \rho_0 &\to \rho_0{}^m \end{aligned} \quad (5.4)$$

The reason for this simple transformation is that the survivability in each case always appears as a simple product of powers of these two parameters. (In some cases the probability is 1, which can be considered as the 0th power of the parameters. In that case we have $1^m = 1$.) This will also be the case for the more general models. Given that we have found the explicit expressions for the more general cases of the model, the generalisation of the moments becomes trivial through the following transformations:

$$\begin{aligned} &\rho_0 \to \rho_0{}^m \\ &\rho_n{}' \to \rho_n{}'^m \\ &\rho_r{}' \to \rho_r{}'^m \end{aligned} \tag{5.5}$$

To find the mth statistical moment for the model developed in chapter 3.1, including a non-ideal MAWS, a complete last salvo and constant missile attack and false alarm probabilities per unit time, we can transform equation (3.13) using equation (5.5). The mth moment is then given as:

$$\begin{aligned} \left\langle P_{survive}{}^m \right\rangle = &\left(\frac{\rho'_n{}^m \gamma_{MA} + \gamma_{FA}}{\rho_0{}^m \gamma_{MA} + \gamma_{FA}} \right)^S e^{\left(\rho_0{}^m - 1\right)\gamma_{MA} t} \\ &+ \left(\rho'_n{}^m \gamma_{MA} + \gamma_{FA}\right)^S e^{-(\gamma_{MA}+\gamma_{FA})t} \left(\sum_{u=0}^{S-1} \frac{t^u}{u!} \left(\left(\rho'_n{}^m \gamma_{MA} + \gamma_{FA}\right)^{u-S} - \left(\rho_0{}^m \gamma_{MA} + \gamma_{FA}\right)^{u-S} \right) \right) \end{aligned} \tag{5.6}$$

$$\rho'_n = \rho_0 + p_D p_{CME,n} \left(1 - \rho_0\right)$$

Likewise, the mth moment for the general model developed in chapter 3.2, which in addition to the previous opens up for variable missile attack and false alarm probabilities per unit time, as well as an incomplete last salvo. Modifying equation (3.28) using the transformation in (5.5), we get:

$$\begin{aligned} \left\langle P_{survive}{}^m \right\rangle = &e^{-\gamma_0 \left(T_{MA}(t) + T_{FA}(t)\right)} \left(\sum_{u=0}^{S-1} \frac{\left(\gamma_0 T_{FA}(t) + \rho'_n{}^m \gamma_0 T_{MA}(t)\right)^u}{u!} \right) \\ &+ \frac{\gamma_0{}^{S-1} e^{\left(\rho_0{}^m - 1\right)\gamma_0 T_{MA}(t)}}{(S-1)!} \int_0^t dt_s \left[e^{\gamma_0 \left(T_{FA}(t_s) - \rho_0{}^m T_{MA}(t_s)\right)} \left(\rho'_r{}^m \gamma_{MA}(t_s) + \gamma_{FA}(t_s)\right) \left(\rho'_n{}^m T_{MA}(t_s) + T_{FA}(t_s)\right)^{S-1} \right] \end{aligned} \tag{5.7}$$

$$\begin{aligned} \rho'_n &= \rho_0 + p_D p_{CME,n} \left(1 - \rho_0\right) \\ \rho'_r &= \rho_0 + p_D p_{CME,r} \left(1 - \rho_0\right) \end{aligned}$$

Using these statistical moments, it is possible to derive different expressions for various statistical properties for the probability distribution, such as the standard deviation, skewness, kurtosis et cetera.

5.2 Mission survival probability as a probability distribution

In the previous chapter we found the statistical moments of the probability distribution. The 1st and 2nd moments give some information about the expectance value and the uncertainty. We will now consider the variation in the mission survival probability as a probability distribution, written as:

$$p\left(x,P\left(t\right)\right) \tag{5.8}$$

Here x is in the range from 0 to 1 (as this range contains all possible values of a probability). The properties of this probability distribution are:

$$\begin{aligned}
&\int_0^1 p\left(x,P\left(t\right)\right)dx = 1\\
&\int_0^1 p\left(x,P\left(t\right)\right)x\,dx = \left\langle P_{Survive}\right\rangle = P\left(t\right)\\
&\int_0^1 p\left(x,P\left(t\right)\right)x^m dx = \left\langle {P_{Survive}}^m\right\rangle
\end{aligned} \tag{5.9}$$

In principle, it is possible to use all the statistical moments to find the probability distribution density from these. This can be done by generating a function which has identical statistical moments as the distribution in question, for example based on a Taylor series given by:

$$g\left(x\right) = \sum_{u=0}^{U} g_u x^u \ , \tag{5.10}$$

where U is arbitrarily large, and equal to the number of moments we wish to include. The 0th moment is 1, as we are considering a probability distribution density. By inserting for the moments, we get:

$$\int_0^1 g\left(x\right)x^m dx = \left\langle {P_{Survive}}^m\right\rangle \tag{5.11}$$

Inserting the expression (5.10) into equation (5.11) and integrates, we get:

$$\int_0^1 \sum_{u=0}^{U} g_u x^{u+m} dx = \left[\sum_{u=0}^{U} \frac{g_u}{u+m+1} x^{u+m+1}\right]_0^1 = \sum_{u=0}^{U} \frac{g_u}{u+m+1} = \left\langle {P_{Survive}}^m\right\rangle \tag{5.12}$$

We then have a linear set of equations with U+1 unknown to solve. To get a solvable equation one should determine all statistical moments from the 0th to the Uth. Defining the following two U+1-dimensional vectors and the matrix **T** of size (U+1)×(U+1):

$$\overrightarrow{\mathbf{G_U}} = \begin{pmatrix} g_0 \\ g_1 \\ \dots \\ g_U \end{pmatrix}, \quad \overrightarrow{\mathbf{P_U}} = \begin{pmatrix} \left\langle P_{Survive}{}^{0} \right\rangle \\ \left\langle P_{Survive}{}^{1} \right\rangle \\ \dots \\ \left\langle P_{Survive}{}^{U} \right\rangle \end{pmatrix}, \quad \mathbf{T} = \begin{bmatrix} T_{0,0} & T_{0,1} & \dots & T_{0,U} \\ T_{1,0} & T_{1,1} & \dots & T_{1,U} \\ \dots & \dots & \dots & \dots \\ T_{U,0} & T_{U,1} & \dots & T_{U,U} \end{bmatrix}, \quad T_{u,m} = \frac{1}{u+m+1}, \qquad (5.13)$$

we can rewrite equation (5.12) as:

$$\mathbf{T}\overrightarrow{\mathbf{G_U}} = \overrightarrow{\mathbf{P_U}} \qquad (5.14)$$

We then see that we can find an expression for *g(x)* by inverting **T**:

$$\overrightarrow{\mathbf{G_U}} = \mathbf{T}^{-1}\overrightarrow{\mathbf{P_U}} \qquad (5.15)$$

5.3 Survival probability of a contingency

We now wish to apply the expression obtained on a contingency deployed for an extended period of time. We make the following assumptions:

- A total of *M* missions are flown with *N* aircraft.
- The mission survival probability is given as *P*, and is for simplicity assumed to be constant.

It is then possible to compare the missions with a weighted coin flip experiment: During a mission there is a probability *P* for the aircraft to return safely, and a probability 1-*P* for it not to do so. The probability distribution of returning from the deployment with *K* aircraft is then given as a binominal distribution:

$$p(K,M,N,P) = \binom{M}{N-K} P^{N-K} (1-P)^{M-N+K}, \quad K = 1,2,\dots,N \qquad (5.16)$$

The probability for returning without any aircraft is given as the complement of the other probabilities. Here, the number of lost aircraft is given as *N-K*.

If the number of missions *M* becomes large, expression (5.16) becomes difficult to calculate with a useful precision. One alternative may be the use of the Poisson distribution, which is the limit of the binominal distribution for large values of *M*. We then have:

$$p(K,M,N,P) = \frac{e^{-(1-P)M}\left((1-P)M\right)^{N-K}}{(N-K)!}, \quad K = 1,2,\dots,N \qquad (5.17)$$

6 NUMERICAL EXAMPLES

We will in this chapter illustrate some effects of the model. We now assume the following parameters:

- False alarm rate: γ_{FA} = 0.5 hr^{-1}, assumed constant during the mission.
- MAWS detection probability of MANPADS: P_D = 0.97.
- Missile attack probability without the use of countermeasures: ρ_0 = 0.1
- Number of potential missiles in scenario: >400.
- Flare dispenser capacity: N = 30.
- Missile attack rate: $\gamma_{MA} = 5\times10^{-4}$ hr^{-1}, assumed to be constant during the mission. All missile attacks are assumed to be independent.

These figures are deliberately unclassified, but they might correspond to a scenario with an intensity corresponding to scenarios in Afghanistan or Iraq. **It is stressed that no conclusions should be drawn about operational behaviour or changes to be made thereof on the basis of the figures found in these numerical examples, as they are just meant to illustrate the theory.**
We will in the following assume that a fixed number of flares are used per salvo during a scenario.

The number of missiles in the scenario is large, and assuming all missile firings are independent, the conditions for the use of Poisson statistics for missile attacks are satisfied.

6.1 Scenario 1

We will in this scenario consider the effect of 5 different flare programs, each with its own (hypothetical) flare decoy efficiency. Furthermore, we assume a mission time duration t of 5 hours. Inserting into the formulas, we have:

Program size	1	2	5	15	30
Flare decoy efficiency	0.5	0.75	0.9	0.95	1.0
$\left(\dfrac{\gamma_{FA}+\rho'_n\gamma_{MA}}{\gamma_{FA}+\rho_0\gamma_{MA}}\right)^S$	1.0131	1.0099	1.0047	1.0017	1.0009
Mission survival probability (%)	99.884	99.939	99.970	99.910	99.855
Mission shoot-down probability (%)	0.146	0.061	0.030	0.090	0.145
Program ranking	5	2	1	3	4

Table 6.1 Table for mission survival probability for various (hypothetical) flare program sizes.

We see from this table that even a perfect program may not have the best overall performance. Furthermore we see that the test criterion expression used for comparison in chapter 4.1.2

decreases with salvo size, meaning that a different mission duration might lead to another ranking.

6.2 Scenario 2

We will in the two following scenarios consider the “empty dispenser” effect on the requirements for the value of the flare decoy efficiency.

We first consider a “short” mission time of 2 hours, which could correspond to a deep strike mission. We will compare flare programs containing 1, 2, 3, 5, 6, 10, 15 and 30 flares. Assuming that the flare decoy efficiency for a single flare program is $\rho_{CME,1}$=0.5, we will now find the required values of the flare decoy efficiency $\rho_{CME,n}$ for other values of n to get equal mission survival probabilities. The flare decoy efficiency is a parameter that can be estimated experimentally through measurements made during classical countermeasures trials, even though the weighting of the various threats in the scenario necessarily become scenario specific.
We then assume that $\rho_{CME,1}$ = 0.5. With the given assumptions, we now investigate what are the required values of $\rho_{CME,2}$, $\rho_{CME,3}$, ρ_{CME5}, $\rho_{CME,6}$, $\rho_{CME,10}$, $\rho_{CME,15}$, and $\rho_{CME,30}$, in order to obtain the same value for the total survival probability as with a single flare program. For simplicity, only salvo sizes divisible by 30 are taken into account.

Using the assumptions made above, it is possible to use the model developed in chapter 3.1. We can then use equation (3.13):

$$P = \left(\frac{\rho'_n \gamma_{MA} + \gamma_{FA}}{\rho_0 \gamma_{MA} + \gamma_{FA}} \right)^S e^{(\rho_0 - 1)\gamma_{MA} t} + \left(\rho'_n \gamma_{MA} + \gamma_{FA}\right)^S e^{-(\gamma_{MA}+\gamma_{FA})t} \left(\sum_{u=0}^{S-1} \frac{t^u}{u!} \left(\left(\rho'_n \gamma_{MA} + \gamma_{FA}\right)^{u-S} - \left(\rho_0 \gamma_{MA} + \gamma_{FA}\right)^{u-S} \right) \right)$$

$$\rho'_n = \rho_0 + p_D p_{CME,n} (1 - \rho_0)$$

Inserting for $n=1$, $\rho_{CME,1}$ = 0.5, we obtain P = 0.99953661. The precision is artificially high for a practical case, but is included to illustrate the perturbative effect of the changes. In practice, this figure means that, on the average, every 2000 mission results in a shoot-down.

To obtain the same values for P with the other values of n (with the same number of significant digits), we need the following values for the flare decoy efficiency:

n = 2: $\rho_{CME,2}$ = 0.5000
n = 3: $\rho_{CME,3}$ = 0.5000
n = 5: $\rho_{CME,5}$ = 0.5015
n = 6: $\rho_{CME,6}$ = 0.5057
n = 10: $\rho_{CME,10}$ = 0.5612
n = 15: $\rho_{CME,5}$ = 0.6856
n = 30: $\rho_{CME,6}$ = 1.1569 : Impossible!

The requirements for the flare decoy efficiency is found to increase significantly only for n=10. It is found that a 30 flare salvo requires a flare decoy efficiency superior to 1, which is impossible. It is therefore not possible to obtain the same mission survival probability for this value. The reason for this is that the reduced number of salvos available becomes a critical limitation for overall system performance.

We now make a note of the critical time duration t_c found in chapter 4.1. We can here use the simplified form $t_c=S/\gamma_{FA}$, and we see that if the critical time is set to be equal to the mission time, this corresponds to $S=\gamma_{FA}t=1$, a salvo count corresponding to n = 30.

6.3 Scenario 3

We now repeat the same exercise as in 2, but now with a mission time duration t = 7 hours. This might be typical of a surveillance scenario.

Inserting for $n=1$ and $\rho_{CME,1}$ = 0.5, we obtain P = 0.9983791. This corresponds to an average shoot-down every 600 mission. To get the same value for P with other values of n (with the same number of significant digits), we require the following values for the flare decoy efficiency:

n = 2: $\rho_{CME,2}$ = 0.5003
n = 3: $\rho_{CME,3}$ = 0.5149
n = 5: $\rho_{CME,5}$ = 0.6475
n = 6: $\rho_{CME,6}$ = 0.7437
n = 10: $\rho_{CME,10}$ = 1.1819 : Impossible!
n = 15: $\rho_{CME,5}$ = 1.7574 : Impossible!
n = 30: $\rho_{CME,6}$ = 3.5072 : Impossible!

With a 7 hour mission time duration, the requirements starts increasing significantly at n=5, and it is not possible to obtain the same mission survival probability with a salvo size equal or superior to 10. If we require the critical time duration to equal the mission time duration, this corresponds to a number of salvos $S = \gamma_{FA}t = 3.5$, corresponding to $n \approx 9$.

These two numerical examples <u>could</u> indicate that the critical time duration defines a limiting value of how large false alarm rates could be tolerated in a defensive aids suite, for a given kind of mission type.

6.4 Scenario 4

We now consider a contingency that is to be deployed abroad for a time period of 3 months. We assume a total of 6 aircraft are deployed, and 2 hour missions are flown 6 times each day, with the total number of missions expected to be 560 for the whole deployment. First assuming that a single flare salvo with a flare decoy efficiency of 0.5 and all other parameters as in scenario 1, leading to a mission survival probability of 0.99954. Inserting into equation (5.17), we find the following probability of losses:

Number of aircraft lost	Probability
0	77.3 %
1	19.9 %
2	2.6 %
3	0.2 %
4	0.0 %
5	0.0 %
6	0.0 %

Table 6.2 Probability of losing aircraft during a 3-months deployment using single flare salvos, with a flare decoy efficiency of 0.5.

We now consider the use of 5-flares-salvos, with a flare decoy efficiency of 0.9, and with the other parameters as specified above. We then get a mission survival probability of 0.99984. Inserting into equation (5.17), we find the following probability of losses:

Number of aircraft lost	Probability
0	91.4
1	8.2
2	0.4
3	0.0
4	0.0
5	0.0
6	0.0

Table 6.3 Probability of losing aircraft during a 3-months deployment using 5-flares-salvos, with a flare decoy efficiency of 0.9.

7 NUMERICAL SIMULATIONS

A series of numerical simulations have been run through to verify the model. Some of these are presented here as an illustration. As numerical input values, we here assume the following:

- False alarm rate: $\gamma_{FA} = 0.5$ hr^{-1}, assumed to be constant during the mission.
- Missile detection probability for MANPADS: $p_D = 0.98$.
- Single missile attack survival probability without use of countermeasures: $\rho_0 = 0.2$
- Single missile attack survival probability with use of countermeasures:
  - 2-flare salvo: $\rho'_2 = 0.7$
  - 5-flare salvo: $\rho'_5 = 0.9$
- Assumed number of missiles in theatre: >400.
- Dispenser capacity: $N = 30$.
- Expected missile attack rate:
  - Either a constant rate $\gamma_{MA} = 5\times10^{-2}$ hr^{-1}, for the entire mission.
  - Or a variable rate, with:
    - $0 \le t \le 2$ hrs: $\gamma_{MA} = 0.5$ hr^{-1},
    - 10 hrs $\le t \le$ 11 hrs: $\gamma_{MA} = 1$ hr^{-1},
    - Otherwise: $\gamma_{MA} = 5\times10^{-2}$ hr^{-1},

During the simulation, a relatively high missile attack rate has been used to reduce the number of simulations required to reduce the statistical noise.

The simulations have been performed as follows: Each mission has been separated into small time intervals of duration $dt = 10^{-3}$ hr. In each interval a number X is produced by a random number generator with a uniform distribution between 0 and 1. If X is smaller than $dt \times \gamma_{MA}$, a missile attack is assumed to have occurred, and a calculation based on a new random number is then made to see if the aircraft survived the missile attack. On survival the remaining flare capacity is updated. If no missile attack occurred, a check is made of whether or not a false alarm was declared by checking if X is larger than $1\text{-}dt \times (\gamma_{FA})$. In the case of a false alarm, the remaining flare capacity is updated.

A sequence of numbers is generated for the whole scenario, the length of which equalling the number of intervals. The sequence values are set to 1 in time intervals where the aircraft has survived, and 0 in the cases where the aircraft has been shot down. If the number of experiments is large, the average value of each interval, averaged over all the experiments approaches the expectance value of the model, within the statistical uncertainty.

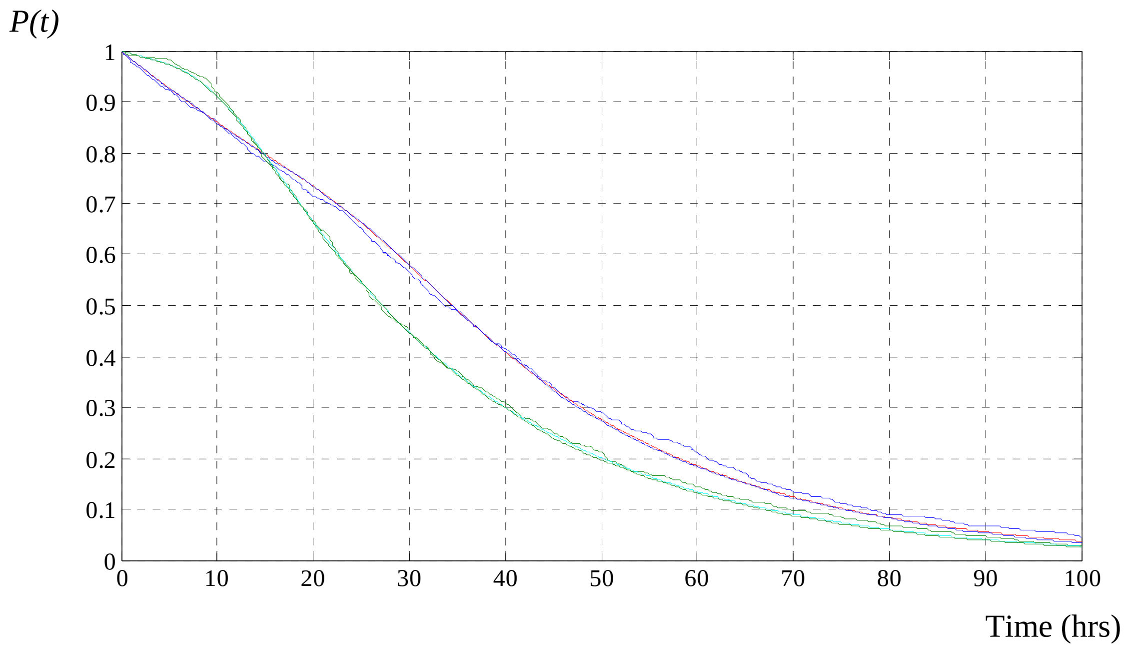

Figure 7.1 Mission survival probability as a function of mission time calculated using the model for 2-flare salvos and 5-flare salvos (red and green curves, respectively), given a uniform missile attack probability, and corresponding simulations averaged over 1000 and 10000 runs. The blue curves corresponds to the 2-flare salvos simulations, and the olive green curves corresponds to the 5-flare salvos simulations. The curves averaged over 10000 runs are found to closely follow the model-based curves.

Figure 7.1 shows a comparison between the mission survival probability as a function of time as obtained by the model and as a function of simulations averaged over 1000 and 10000 runs, respectively. This is done for both flare programs described above. As can be seen, there is a good agreement between simulations and the model. It is especially interesting to note the extra "hump" occurring on the red curve for the 2-flare model at $t \approx 25$ hrs.

Figure 7.2 shows a comparison between the modelled mission survival probability as a function of time and corresponding simulations averaged over 1000 and 10000 runs for the two previously mentioned flare programs, but this time with an increased expected missile attack rates in the 1-2 hr region and the 10-11 hr region.

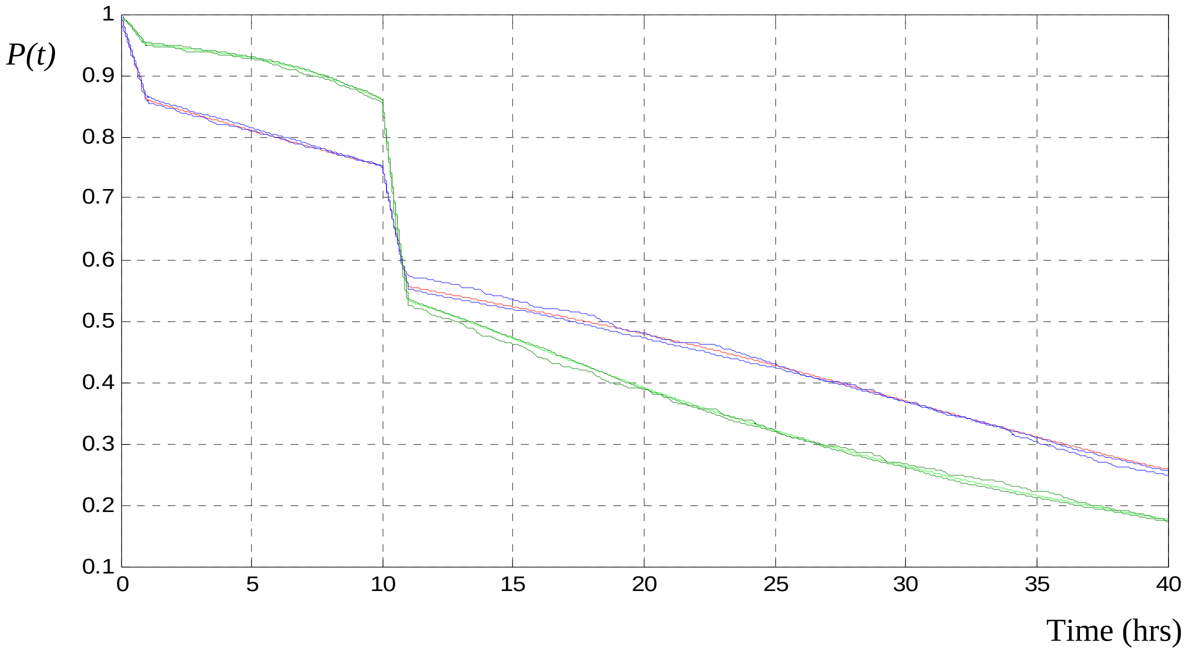

Figure 7.2 *Model predicted mission survival probability as a function of mission time, for 2- and 5-flare programs (red and green curve), and corresponding simulations averaged over 1000 and 10000 runs, respectively. The blue curves corresponds to the 2-flare salvos simulations, and the olive green curves corresponds to the 5-flare salvos simulations. The 10000 run simulations are found to closely follow the model curves in detail.*

8 LIMITATIONS OF THE GENERAL MODEL

A way to find immediate limitations in the model, is to look at the assumptions made:

1. **All missile attacks are independent.** This might be partly correct, as this kind of system is coordinated on a relatively low level of command. Natural clustering around strategic points could be included in the mission profile, but clustering around unknown strategic points might break with the model properties. An update of the mission profile could, at least partially, compensate for this. Co-ordinated attacks on a platform could nevertheless represent a problem, especially in co-ordinated air defences, or in scenarios where for example VIP platforms are being attacked.
2. **There is a large number (in principle an infinite amount) of missiles in the theatre.** Statistically speaking on the order of 100 missiles or more could be well represented by a Poisson distribution, whereas the expected distribution of attacks might be better

represented by a binomial probability distribution for small missile numbers (<10). It should be emphasised that not all missiles need to be employed during the mission; it is sufficient that they represent a potential threat. The solution using a binomial distribution is readily found by inserting this distribution into the general expression found earlier. A more significant issue is the distribution of a limited number of one missile type (for example modern/expensive systems), but with a certain abundance of some cheaper and less efficient missile type. This might affect the model. One solution to the problem might be a dynamic update of the system parameters. Unfortunately, such an implementation appears unrealistic.

3. **There is *à priori* knowledge about the missile attack and the false alarm probability per unit time.** It is of course not possible to predict the exact timing of the missile attacks or of the false alarms. The expected false alarm rate is largely predictable, due to the fact that it is noise based, and that the noise processes are fairly well understood. Equally important, it is possible to measure noise levels during flight, enabling a relatively good false alarm rate estimate. The picture is much more complicated for the missile attack rate because there are several complicating factors, many of those being difficult to estimate. Furthermore, the very low occurrence of missile attacks makes it difficult to validate a model of the missile attack rate estimate.
4. **There is a fixed missile attack survival probability given that one uses a flare program with a given number of flares.** Inserting an average value of a parameter into a model is always problematic if the expression is a non-linear function of the parameter. It is possible, though, to replace the fixed missile attack survival probabilities by a time dependent function if more information about the threat scenario is available. It is also possible to split the threat into several cases as mentioned in chapter 3.3. If the threat types are sufficiently split up, it is possible to keep all parameters other than the missile attack rates constant.

In addition to this, there are other limitations that have not been included:

5. **The possibility of multiple flare types in the dispensers.** This requires an additional extension of the model according to the same principles used earlier, with times t_{s1}, t_{s2} etc. where the last flares of each type are being spent.
6. **Possibility of misfire.** This corresponds to extra flare consumption, and will depend on the systems ability to cope with the issue or not. If the system is not able to handle misfire (by launching a new flare), the effect of misfire could be included as a decreased effective missile attack survival probability.
7. **The possibility of multiple incomplete last salvos.** This might be beneficial if there are marginal differences between small salvo and medium salvo performance, for example if a certain missile in the theatre requires a large number of flares to be decoyed, whereas all the other missile types are efficiently handled using single flare programs. This complicates the model, as it may be necessary to handle not only one last salvo, but several salvos, complicating the summation terms.

One more element worth consideration is the fact that the model assumes the use of a fixed program size throughout the mission. This may not necessarily be the case, especially if any one of the missile rate(s) and the false alarm rate varies as a function of time. As an example, one might imagine an extreme case in which the missile attack rate is (very close to) zero. It would then be interesting to have a small amount of flares being deployed to minimise the risk of depleting the dispensers. The model is nevertheless well suited for systems for which it is not possible to modify the countermeasures settings of either the MAWS or the flare programs during flight. It is also possible during flight to change flare programs while on mission. This is generally possible with most equipment used these days.

It might seem intuitively possible and even probable that the best protection in case of a constant missile attack and false alarm probability per unit time is achieved with a fixed program size throughout the mission. This nevertheless requires verification. In the model presented here, it has been presumed that input parameters are kept constant, even though events (or the lack of events) could make the continued updating of the input parameters beneficial. In addition to a sequence of salvos with sizes of type {5, 5, 5, 5, 5, 5}, it is for instance possible to imagine salvos with varying sizes, such as salvos with sizes of type {8, 7, 6, 4, 3, 2}. This is expected to complicate things significantly, as there is no purely mathematical description of the salvo performance as a function of salvo size.

9 ALTERNATIVE MODELS

In this work one approach has been chosen to develop the mathematical model. It can be mentioned that there are also other ways to develop models, some of which have been tried. Generally, though, these other methods have limitations or are difficult to establish in a mathematical form suitable for practical evaluation. Some of the models could nevertheless give additional insight into the model behaviour, and therefore we will outline how to establish at least one of these solutions, in which a non-ideal detection probability and constant expected missile attack and false alarm rates are assumed.

One way to look at the problem is to say that within the mission duration one might have i missile attacks and j false alarms, where both i and j can range between 0 and ∞. The probability of this combination to occur is, as before, given as the product of two Poisson probability distributions. In the case where $i + j \leq S$, the survival probability of this category does not depend upon the sequence of false alarms and missile attacks, and we get the same result as was developed in chapter 3.1. If, however, $i + j > S$, the sequence of events is of importance. As a Poisson distribution has been assumed for both the missile attacks and the false alarms, any possible sequence is equally likely. During the first S events, there will be countermeasures to deploy, and the probability of surviving each missile attack is given by ρ'_n. During the subsequent attacks, the probability of surviving each missile attack is given as ρ_0, as there are no countermeasures left. We are therefore interested in the probability of finding the probability that there are k missile attacks among the first S events, given that there are a total of i missile attacks and j false alarms during the mission time. This probability is equal to the probability that *k* out of *S* balls drawn out of a box containing *i* red and *j* white balls, are red. This probability is found to be:

$$P(k,S,i,j)=\frac{i!\,j!(i+j-S)!S!}{(i-k)!(j+k-S)!(i+j)!(k)!(S-k)!}$$

To continue this development, it is important to recall that k ranges from 0 and min(*S*,*i*). The total expression of the mission survival probability becomes fairly complicated, and since we already have a useful expression, we will not pursue this model any further.

10 CONCLUSION

This report presents a general model to quantitatively determine the mission survival probability of an aircraft equipped with a limited amount of expendable countermeasures, and also contains several versions of closed form equations of the actual mission survival probability. The model can be used to optimise central parameters in the defensive aids suite, and can be adapted to mission specific characteristics. The use of the model enables a dynamic update of the defensive aids suite, thus enabling the benefit of added information on events (or lack of events) or previous events during the mission. The model also enables a dynamic optimisation of the missile approach warning system properties. The model gives a significant advantage over Monte-Carlo-simulations through an increased understanding of the problem, a better interpretation of results and an efficient optimisation of parameters, especially given the significant parameter uncertainty. The model could be employed to optimise overall platform survivability in a quantitative manner, and the efficiency enables a dynamic optimisation over a large parameter space. One consequence of the model is that the optimisation of the defensive aids suite becomes a natural part of pre-flight mission planning.

A bonus appears from the equations developed in that they become a tool for optimisation of missile warning key parameters. By performing in-flight noise measurements in the missile warning system, it is possible to estimate the false alarm rate and the detection probability as a function of the noise threshold, and choose a mission specific optimum detection threshold dynamically. This important conclusion is expected to significantly increase the capability of the defensive aids suite.

The quantitative aspects of the model mean that it is possible to use it as a tool for cost analysis. One application can be the investigation of the effect of upgrading certain components of the system, for example to compare the cost and effect of changing the missile approach warning system versus installing additional dispensers. It is also possible to consider which platform (for example helicopter versus fast jet) upgrades will produce the highest benefit for the Air Force as a whole.

Chapter 3.3 and 5 deal with uncertainty, in terms of input parameter uncertainty as well as uncertainty related to statistical variations. It is highly recommended that any use of the model on real life scenarios include studies of the uncertainty with regards to input parameters as well as estimates of possible statistical variations, in order to obtain the best possible overview of the situation. The errors caused by inaccuracies might lead to unacceptable risk during a mission.

One point that has not been commented upon is the fact that the model predicts that it could be of overall benefit to increase the risk of emptying the dispensers during a mission. This is fairly counterintuitive, and in parts of the EW community the flare programs have deliberately been minimised in order to avoid the risk of flying parts of the mission with empty dispensers. The model nevertheless indicates that the risk of flying parts of the mission with empty dispensers can be outweighed by the benefit of a higher performance of each flare program. A completely different issue is the fact that this may increase the overall cost of the EW defensive aids suite, a performance that could have been countered for example by buying a better performing missile approach warning system. By its quantitative nature, the model actually enables these kind of calculations (best protection for a given sum of money), making it possible to compare different acquisition strategies. It may also be possible to improve the prioritisation of countermeasures enhancements and upgrades of different components, also between platforms. Here, threat exposure and choices of what kind of international (or national) engagements to participate in will be decisive.

10.1 Further work

The greatest challenge will be to quantify the missile attack probability per unit time, as it is very small, very uncertain and difficult to assess. In order to use this model in practice, a well founded model should be established to determine this parameter in particular. A basis should be detailed intelligence data coupled with historical data, together with a detailed model taking into account various factors such as platform exposure, terrain and terrain features, infrastructure, strategic elements (airfields, headquarters,...), and known strategies and tactics.

Another issue is that the equation assumes no coupling between individual missile attacks, although a variable missile attack rate can compensate for this to a certain degree. One important example here is the multi missile co-ordinated attack. Investigations should therefore be made to examine the effect of connections between missile attacks.

In the model presented here, it has been assumed that à priori flare program sizes are kept fixed during a mission, even though events (or lack of events) leads to a change in program setting and missile warning parameters dynamically due to a continuous update of input parameters, the most important of these being the remaining mission time duration. Investigations should be initiated to see whether a sequence of flare programs with variable salvo sizes is beneficial. The most important case of this study should be the optimisation of flare program size as a function of variations in the expected ratio between the expected missile attack rate and the false alarm rate.

11 ACKNOWLEDGEMENTS

Several persons have contributed to this report through proof reading, discussions, problem solving, encouragement, constructive criticism and practical guesstimates. I would like to send a special thank you to Jonas Moen, Ingebjørg Kåsen, Stein Kristoffersen, Arne Petter Bartholsen, Tor Holmboe, Eivind Engebretsen, Tor-Odd Høydal and other members of the EW group at FFI, as well as Major Øystein Bringe at the RNoAF EW Support Centre and Lieutenant Colonel Tom-Ivar Punsvik at the EW Branch of the RNoAF Air Staff, Inspectorate of Air Operations.

APPENDIX

A DEVELOPMENT OF THE EXPRESSION OF THE MISSION SURVIVAL PROBABILITY FOUND IN CHAPTER 3.1

We will here go through the calculations in the case where we have a non-perfect missile approach warning system, and constant missile attack and false alarm probabilities per unit time. Starting with the model developed in chapter 3.1, we have:

$$P = \left(\sum_{i=0}^{S-1} \sum_{j=0}^{S-1-i} p_{MA,i}(t)\, \rho'^{\,i}_{n}\, p_{FA,j}(t) \right) + \left(\int_0^t \sum_{i=0}^{S-1} \left(dt_s \left(\rho'_n \gamma_{MA} + \gamma_{FA} \right) \left(p_{MA,i}(t_s)\, \rho'^{\,i}_{n}\, p_{FA,S-1-i}(t_s) \right) \sum_{k=0}^{\infty} \left(p_k(t-t_s)\, \rho_0^{\,k} \right) \right) \right) \tag{A.1}$$

Inserting the probability distributions, we get:

$$P = \left(\sum_{i=0}^{S-1} \sum_{j=0}^{S-1-i} \frac{e^{-\gamma_{MA} t} (\gamma_{MA} t)^i}{i!} \rho'^{\,i}_{n} \frac{e^{-\gamma_{FA} t} (\gamma_{FA} t)^j}{j!} \right) + \int_0^t \sum_{i=0}^{S-1} dt_s \left(\rho'_n \gamma_{MA} + \gamma_{FA} \right) \left(\frac{e^{-\gamma_{MA} t_s} (\gamma_{MA} t_s)^i}{i!} \rho'^{\,i}_{n} \frac{e^{-\gamma_{FA} t_s} (\gamma_{FA} t_s)^{S-1-i}}{(S-1-i)!} \right) \sum_{k=0}^{\infty} \frac{e^{-\gamma_{MA}(t-t_s)} \left(\gamma_{MA}(t-t_s) \right)^k}{k!} \rho_0^{\,k} \tag{A.2}$$

The last summation is simplified by collecting all terms with exponent *k*, multiplying with a unit ratio, and move some terms out of the sum:

$$P = \left(\sum_{i=0}^{S-1} \sum_{j=0}^{S-1-i} \frac{e^{-\gamma_{MA} t} (\gamma_{MA} t)^i}{i!} \rho'^{\,i}_{n} \frac{e^{-\gamma_{FA} t} (\gamma_{FA} t)^j}{j!} \right) + \int_0^t \sum_{i=0}^{S-1} \left[dt_s \left(\rho'_n \gamma_{MA} + \gamma_{FA} \right) \left(\frac{e^{-\gamma_{MA} t_s} (\gamma_{MA} t_s)^i}{i!} \rho'^{\,i}_{n} \frac{e^{-\gamma_{FA} t_s} (\gamma_{FA} t_s)^{S-1-i}}{(S-1-i)!} \right) \frac{e^{-\gamma_{MA}(t-t_s)}}{e^{-\rho_0 \gamma_{MA}(t-t_s)}} \times \sum_{k=0}^{\infty} \frac{e^{-\rho_0 \gamma_{MA}(t-t_s)} \left(\rho_0 \gamma_{MA}(t-t_s) \right)^k}{k!} \right] \tag{A.3}$$

The summation is now recognisable as a sum of all terms of a Poisson distribution, a sum that equals 1. We can also simplify the exponential ratio:

$$P=\left(\sum_{i=0}^{S-1}\sum_{j=0}^{S-1-i}\frac{e^{-\gamma_{MA}t}\left(\gamma_{MA}t\right)^{i}}{i!}\rho'_{n}{}^{i}\frac{e^{-\gamma_{FA}t}\left(\gamma_{FA}t\right)^{j}}{j!}\right)$$
$$+\left(\int_{0}^{t}\sum_{i=0}^{S-1}\left(dt_s\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)\left(\frac{e^{-\gamma_{MA}t_s}\left(\gamma_{MA}t_s\right)^{i}}{i!}\rho'_{n}{}^{i}\frac{e^{-\gamma_{FA}t_s}\left(\gamma_{FA}t_s\right)^{S-1-i}}{\left(S-1-i\right)!}\right)e^{\left(\rho_0-1\right)\gamma_{MA}\left(t-t_s\right)}\right)\right) \quad \text{(A.4)}$$

We now remove movable factors out of the sum, and collects factors with common exponent *i*:

$$P=\left(\sum_{i=0}^{S-1}\sum_{j=0}^{S-1-i}\frac{e^{-\gamma_{MA}t}\left(\gamma_{MA}t\right)^{i}}{i!}\rho'_{n}{}^{i}\frac{e^{-\gamma_{FA}t}\left(\gamma_{FA}t\right)^{j}}{j!}\right)$$
$$+\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)\int_{0}^{t}\left(dt_st_s{}^{S-1}e^{-\gamma_{MA}t_s}e^{-\gamma_{FA}t_s}e^{\left(\rho_0-1\right)\gamma_{MA}\left(t-t_s\right)}\sum_{i=0}^{S-1}\left(\frac{1}{i!\left(S-1-i\right)!}\left(\rho'_n\gamma_{MA}\right)^{i}\left(\gamma_{FA}\right)^{S-1-i}\right)\right)\right) \quad \text{(A.5)}$$

We then multiply with a unit ratio with (*S*-1)! in both numerator and denominator, moving the denominator outside the sum:

$$P=\left(\sum_{i=0}^{S-1}\sum_{j=0}^{S-1-i}\frac{e^{-\gamma_{MA}t}\left(\gamma_{MA}t\right)^{i}}{i!}\rho'_{n}{}^{i}\frac{e^{-\gamma_{FA}t}\left(\gamma_{FA}t\right)^{j}}{j!}\right)$$
$$+\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)\int_{0}^{t}dt_st_s{}^{S-1}e^{-\gamma_{MA}t_s}e^{-\gamma_{FA}t_s}e^{\left(\rho_0-1\right)\gamma_{MA}\left(t-t_s\right)}\frac{1}{\left(S-1\right)!}\sum_{i=0}^{S-1}\frac{\left(S-1\right)!}{i!\left(S-1-i\right)!}\left(\rho'_n\gamma_{MA}\right)^{i}\left(\gamma_{FA}\right)^{S-1-i} \quad \text{(A.6)}$$

The sum can now be recognised as a binomial series that can be simplified:

$$P=\left(\sum_{i=0}^{S-1}\sum_{j=0}^{S-1-i}\frac{e^{-\gamma_{MA}t}\left(\gamma_{MA}t\right)^{i}}{i!}\rho'_{n}{}^{i}\frac{e^{-\gamma_{FA}t}\left(\gamma_{FA}t\right)^{j}}{j!}\right)$$
$$+\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)\int_{0}^{t}\left(dt_st_s{}^{S-1}e^{-\gamma_{MA}t_s}e^{-\gamma_{FA}t_s}e^{\left(\rho_0-1\right)\gamma_{MA}\left(t-t_s\right)}\frac{\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{S-1}}{\left(S-1\right)!}\right)\right) \quad \text{(A.7)}$$

We now rearrange the exponentials and move the factors that do not depend on t_s outside the integral:

$$P=\left(\sum_{i=0}^{S-1}\sum_{j=0}^{S-1-i}\frac{e^{-\gamma_{MA}t}\left(\gamma_{MA}t\right)^{i}}{i!}\rho'_{n}{}^{i}\frac{e^{-\gamma_{FA}t}\left(\gamma_{FA}t\right)^{j}}{j!}\right)$$
$$+\left(\frac{\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{S}e^{\left(\rho_0-1\right)\gamma_{MA}t}}{\left(S-1\right)!}\int_{0}^{t}\left(dt_st_s{}^{S-1}e^{-\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)t_s}\right)\right) \quad \text{(A.8)}$$

The next step is a change of variables in the integral. By using $T_s=\left(\rho_0\gamma+\gamma_{FA}\right)t_s$ we get:

$$P=\left(\sum_{i=0}^{S-1}\sum_{j=0}^{S-1-i}\frac{e^{-\gamma_{MA}t}\left(\gamma_{MA}t\right)^i}{i!}\rho'^{\,i}_n\frac{e^{-\gamma_{FA}t}\left(\gamma_{FA}t\right)^j}{j!}\right)+\left(\frac{\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^S e^{(\rho_0-1)\gamma_{MA}t}}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^S(S-1)!}\int_0^{(\rho_0\gamma_{MA}+\gamma_{FA})t}\left(dT_sT_s^{S-1}e^{-T_s}\right)\right) \tag{A.9}$$

We now simplify the first double summation. First, we take the exponentials out of the double summation, and recollect the factors with a common exponent *i*:

$$P=\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\sum_{i=0}^{S-1}\sum_{j=0}^{S-1-i}\frac{\left(\rho'_n\gamma_{MA}t\right)^i}{i!}\frac{\left(\gamma_{FA}t\right)^j}{j!}\right)+\left(\frac{\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^S e^{(\rho_0-1)\gamma_{MA}t}}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^S(S-1)!}\int_0^{(\rho_0\gamma_{MA}+\gamma_{FA})t}\left(dT_sT_s^{S-1}e^{-T_s}\right)\right) \tag{A.10}$$

We then make a change of variables, with *u=i+j* and *v=j* (so that *i=u-v* and *j=v*), and regroup the terms in the double sum. By letting *u* go from 0 to *S*-1 and *v* go from 0 to *u*, all terms have been accounted for. We then get:

$$P=\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\sum_{u=0}^{S-1}\sum_{v=0}^{u}\frac{\left(\rho'_n\gamma_{MA}t\right)^{u-v}}{u-v!}\frac{\left(\gamma_{FA}t\right)^v}{v!}\right)+\left(\frac{\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^S e^{(\rho_0-1)\gamma_{MA}t}}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^S(S-1)!}\int_0^{(\rho_0\gamma_{MA}+\gamma_{FA})t}\left(dT_sT_s^{S-1}e^{-T_s}\right)\right) \tag{A.11}$$

A unit ratio is multiplied with the double sum, and the denominator is moved outside the inner sum. This gives:

$$P=\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\sum_{u=0}^{S-1}\frac{1}{u!}\sum_{v=0}^{u}\frac{u!}{u-v!v!}\left(\rho'_n\gamma_{MA}t\right)^{u-v}\left(\gamma_{FA}t\right)^v\right)+\left(\frac{\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^S e^{(\rho_0-1)\gamma_{MA}t}}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^S(S-1)!}\int_0^{(\rho_0\gamma_{MA}+\gamma_{FA})t}\left(dT_sT_s^{S-1}e^{-T_s}\right)\right) \tag{A.12}$$

It is possible to recognise the inner sum as a binomial series that can be simplified:

$$P=\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\sum_{u=0}^{S-1}\frac{\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)t\right)^u}{u!}\right)+\left(\frac{\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^S e^{(\rho_0-1)\gamma_{MA}t}}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^S(S-1)!}\int_0^{(\rho_0\gamma_{MA}+\gamma_{FA})t}\left(dT_sT_s^{S-1}e^{-T_s}\right)\right) \tag{A.13}$$

In the subsequent development we will use the following mathematical relation:

$$\frac{d}{dx}\left(\sum_{u=0}^{S-1}\frac{e^{-x}x^u}{u!}\right)=-\sum_{u=0}^{S-1}\frac{e^{-x}x^u}{u!}+\sum_{u=1}^{S-1}\frac{e^{-x}ux^{u-1}}{u!}$$
$$=-\sum_{u=0}^{S-1}\frac{e^{-x}x^u}{u!}+\sum_{u=1}^{S-1}\frac{e^{-x}x^{u-1}}{(u-1)!}$$
$$=-\sum_{u=0}^{S-1}\frac{e^{-x}x^u}{u!}+\sum_{u=0}^{S-2}\frac{e^{-x}x^u}{u!} \tag{A.14}$$
$$=-\frac{e^{-x}x^{S-1}}{(S-1)!}$$

Integrating this relation, we obtain:

$$\left[\sum_{u=0}^{S-1}\frac{e^{-x}x^u}{u!}\right]_0^a=\left(\sum_{u=0}^{S-1}\frac{e^{-a}a^u}{u!}\right)-1=-\int_0^a dx\frac{e^{-x}x^{S-1}}{(S-1)!} \tag{A.15}$$

The integral in (A.13) can then be written as a sum:

$$P=\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\sum_{u=0}^{S-1}\frac{\left((\rho'_n\gamma_{MA}+\gamma_{FA})t\right)^u}{u!}\right)$$
$$+\frac{(\rho'_n\gamma_{MA}+\gamma_{FA})^S e^{(\rho_0-1)\gamma_{MA}t}}{(\rho_0\gamma_{MA}+\gamma_{FA})^S}\left(1-\sum_{u=0}^{S-1}\frac{e^{-(\rho_0\gamma_{MA}+\gamma_{FA})t}\left((\rho_0\gamma_{MA}+\gamma_{FA})t\right)^u}{u!}\right) \tag{A.16}$$

Reordering the terms, we finally get:

$$P=\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^S e^{(\rho_0-1)\gamma_{MA}t}$$
$$+(\rho'_n\gamma_{MA}+\gamma_{FA})^S e^{-(\gamma_{MA}+\gamma_{FA})t}\sum_{u=0}^{S-1}\frac{t^u}{u!}\left((\rho'_n\gamma_{MA}+\gamma_{FA})^{u-S}-(\rho_0\gamma_{MA}+\gamma_{FA})^{u-S}\right) \tag{A.17}$$

Alternatively, the expression can be written with the integral form. We then start with:

$$P=\left(e^{-(\gamma_{MA}+\gamma_{FA})t}e^{(\rho'_n\gamma_{MA}+\gamma_{FA})t}\sum_{u=0}^{S-1}\frac{e^{-(\rho'_n\gamma_{MA}+\gamma_{FA})t}\left((\rho'_n\gamma_{MA}+\gamma_{FA})t\right)^u}{u!}\right)$$
$$+\left(\frac{(\rho'_n\gamma_{MA}+\gamma_{FA})^S e^{(\rho_0-1)\gamma_{MA}t}}{(\rho_0\gamma_{MA}+\gamma_{FA})^S(S-1)!}\int_0^{(\rho_0\gamma_{MA}+\gamma_{FA})t}\left(dT_sT_s^{S-1}e^{-T_s}\right)\right) \tag{A.18}$$

Replacing the sum by the integral form, we have:

$$P = \left(e^{(\rho'_n - 1)\gamma_{MA} t} \left(1 - \int_0^{(\rho'_n \gamma_{MA} + \gamma_{FA})t} \frac{dT_s T_s^{S-1} e^{-T_s}}{(S-1)!} \right) \right) + \left(\frac{(\rho'_n \gamma_{MA} + \gamma_{FA})^S e^{(\rho_0 - 1)\gamma_{MA} t}}{(\rho_0 \gamma_{MA} + \gamma_{FA})^S (S-1)!} \int_0^{(\rho_0 \gamma_{MA} + \gamma_{FA})t} \left(dT_s T_s^{S-1} e^{-T_s} \right) \right) \tag{A.19}$$

We now change the integration variable to get common integral limits:

$$P = e^{(\rho'_n - 1)\gamma_{MA} t} + \frac{(\rho'_n \gamma_{MA} + \gamma_{FA})^S e^{(\rho_0 - 1)\gamma_{MA} t}}{(S-1)!} \int_0^t \left(dt_s t_s^{S-1} e^{-(\rho_0 \gamma_{MA} + \gamma_{FA}) t_s} \right) - \frac{(\rho'_n \gamma_{MA} + \gamma_{FA})^S e^{(\rho'_n - 1)\gamma_{MA} t}}{(S-1)!} \int_0^t \left(dt_s t_s^{S-1} e^{-(\rho'_n \gamma_{MA} + \gamma_{FA}) t_s} \right) \tag{A.20}$$

Reordering the terms, we finally get:

$$P = e^{(\rho'_n - 1)\gamma_{MA} t} + \frac{(\rho'_n \gamma_{MA} + \gamma_{FA})^S}{(S-1)!} \int_0^t \left(dt_s t_s^{S-1} \left(e^{(\rho_0 - 1)\gamma_{MA} t - (\rho_0 \gamma_{MA} + \gamma_{FA}) t_s} - e^{(\rho'_n - 1)\gamma_{MA} t - (\rho'_n \gamma_{MA} + \gamma_{FA}) t_s} \right) \right) \tag{A.21}$$

B DEVELOPMENT OF THE GENERAL EXPRESSION FOUND IN CHAPTER 3.2

We here develop the case with variable missile attack and false alarm rates. The derivation is fairly similar to the derivation in appendix A. Starting with the general formula, we have:

$$P = \left(\sum_{i=0}^{S-1} \sum_{j=0}^{S-1-i} p_i\left(T_{MA}(t)\right) \rho'^{\,i}_n \, p_j\left(T_{FA}(t)\right) \right) + \int_0^t dt_s \left(\left(\rho'_r \gamma_{MA}(t_s) + \gamma_{FA}(t_s) \right) \times \left(\sum_{i=0}^{S-1} \left(p_i\left(T_{MA}(t_s)\right) \rho'^{\,i}_n \, p_{S-1-i}\left(T_{FA}(t_s)\right) \right) \sum_{k=0}^{\infty} p_k\left(T_{MA}(t) - T_{MA}(t_s)\right) \rho_0^{\,k} \right) \right) \tag{B.1}$$

Inserting the probability distributions, we get:

$$P = \left(\sum_{i=0}^{S-1} \sum_{j=0}^{S-1-i} \frac{e^{-\gamma_0 T_{MA}(t)} \left(\gamma_0 T_{MA}(t)\right)^i}{i!} \rho'^{\,i}_n \frac{e^{-\gamma_0 T_{FA}(t)} \left(\gamma_0 T_{FA}(t)\right)^j}{j!} \right) + \int_0^t dt_s \left[\left(\rho'_r \gamma_{MA}(t_s) + \gamma_{FA}(t_s) \right) \sum_{i=0}^{S-1} \left(\frac{e^{-\gamma_0 T_{MA}(t_s)} \left(\gamma_0 T_{MA}(t_s)\right)^i}{i!} \rho'^{\,i}_n \frac{e^{-\gamma_0 T_{FA}(t_s)} \left(\gamma_0 T_{FA}(t_s)\right)^{S-1-i}}{(S-1-i)!} \right) \times \sum_{k=0}^{\infty} \frac{e^{-\gamma_0 \left(T_{MA}(t) - T_{MA}(t_s)\right)} \left(\gamma_0 \left(T_{MA}(t) - T_{MA}(t_s)\right)\right)^k}{k!} \rho_0^{\,k} \right] \tag{B.2}$$

In the last sum we collect terms with common exponents and add a unit ratio:

$$
\begin{aligned}
P = & \left(\sum_{i=0}^{S-1} \sum_{j=0}^{S-1-i} \frac{e^{-\gamma_0 T_{MA}(t)} \left(\gamma_0 T_{MA}(t)\right)^i}{i!} \rho'^{\,i}_n \frac{e^{-\gamma_0 T_{FA}(t)} \left(\gamma_0 T_{FA}(t)\right)^j}{j!} \right) \\
& + \int_0^t dt_s \left[\left(\rho'_r \gamma_{MA}(t_s) + \gamma_{FA}(t_s)\right) \sum_{i=0}^{S-1} \left(\frac{e^{-\gamma_0 T_{MA}(t_s)} \left(\gamma_0 T_{MA}(t_s)\right)^i}{i!} \rho'^{\,i}_n \frac{e^{-\gamma_0 T_{FA}(t_s)} \left(\gamma_0 T_{FA}(t_s)\right)^{S-1-i}}{(S-1-i)!} \right) \right. \\
& \qquad \left. \times \sum_{k=0}^{\infty} \frac{e^{-\rho_0 \gamma_0 (T_{MA}(t) - T_{MA}(t_s))}}{e^{-\rho_0 \gamma_0 (T_{MA}(t) - T_{MA}(t_s))}} \frac{e^{-\gamma_0 (T_{MA}(t) - T_{MA}(t_s))} \left(\rho_0 \gamma_0 \left(T_{MA}(t) - T_{MA}(t_s)\right)\right)^k}{k!} \right]
\end{aligned} \tag{B.3}
$$

In the last sum we take out the first factor in the second ratio and the denominator of the first ratio out of the sum:

$$
\begin{aligned}
P = & \left(\sum_{i=0}^{S-1} \sum_{j=0}^{S-1-i} \frac{e^{-\gamma_0 T_{MA}(t)} \left(\gamma_0 T_{MA}(t)\right)^i}{i!} \rho'^{\,i}_n \frac{e^{-\gamma_0 T_{FA}(t)} \left(\gamma_0 T_{FA}(t)\right)^j}{j!} \right) \\
& + \int_0^t dt_s \left[\left(\rho'_r \gamma_{MA}(t_s) + \gamma_{FA}(t_s)\right) \sum_{i=0}^{S-1} \left(\frac{e^{-\gamma_0 T_{MA}(t_s)} \left(\gamma_0 T_{MA}(t_s)\right)^i}{i!} \rho'^{\,i}_n \frac{e^{-\gamma_0 T_{FA}(t_s)} \left(\gamma_0 T_{FA}(t_s)\right)^{S-1-i}}{(S-1-i)!} \right) \right. \\
& \qquad \left. \times \frac{e^{-\gamma_0 (T_{MA}(t) - T_{MA}(t_s))}}{e^{-\rho_0 \gamma_0 (T_{MA}(t) - T_{MA}(t_s))}} \underbrace{\sum_{k=0}^{\infty} \frac{e^{-\rho_0 \gamma_0 (T_{MA}(t) - T_{MA}(t_s))} \left(\rho_0 \gamma_0 \left(T_{MA}(t) - T_{MA}(t_s)\right)\right)^k}{k!}}_{\text{The sum now has the form } \sum_{k=0}^{\infty} \frac{e^{-a} a^k}{k!} = 1} \right]
\end{aligned} \tag{B.4}
$$

The ratio outside the sum can be simplified, as can the sum itself:

$$
\begin{aligned}
P = & \left(\sum_{i=0}^{S-1} \sum_{j=0}^{S-1-i} \frac{e^{-\gamma_0 T_{MA}(t)} \left(\gamma_0 T_{MA}(t)\right)^i}{i!} \rho'^{\,i}_n \frac{e^{-\gamma_0 T_{FA}(t)} \left(\gamma_0 T_{FA}(t)\right)^j}{j!} \right) \\
& + \int_0^t dt_s \left[\left(\rho'_r \gamma_{MA}(t_s) + \gamma_{FA}(t_s)\right) \sum_{i=0}^{S-1} \left(\frac{e^{-\gamma_0 T_{MA}(t_s)} \left(\gamma_0 T_{MA}(t_s)\right)^i}{i!} \rho'^{\,i}_n \frac{e^{-\gamma_0 T_{FA}(t_s)} \left(\gamma_0 T_{FA}(t_s)\right)^{S-1-i}}{(S-1-i)!} \right) \right. \\
& \qquad \left. \times e^{(\rho_0 - 1) \gamma_0 (T_{MA}(t) - T_{MA}(t_s))} \right]
\end{aligned} \tag{B.5}
$$

The sum in line 2 can also be simplified. We start by withdrawing all exponential terms outside the sum and collect all factors with common exponent inside the sum:

$$P=\left(\sum_{i=0}^{S-1}\sum_{j=0}^{S-1-i}\frac{e^{-\gamma_0 T_{MA}(t)}\left(\gamma_0 T_{MA}(t)\right)^i}{i!}\rho'^{\,i}_n\frac{e^{-\gamma_0 T_{FA}(t)}\left(\gamma_0 T_{FA}(t)\right)^j}{j!}\right)$$
$$+\int_0^t dt_s\left[\left(\rho'_r\gamma_{MA}(t_s)+\gamma_{FA}(t_s)\right)e^{-\gamma_0\left(T_{MA}(t_s)+T_{FA}(t_s)\right)}\sum_{i=0}^{S-1}\left(\frac{\left(\rho'_n\gamma_0 T_{MA}(t_s)\right)^i\left(\gamma_0 T_{FA}(t_s)\right)^{S-1-i}}{i!(S-1-i)!}\right)\right. \tag{B.6}$$
$$\left.\times e^{(\rho_0-1)\gamma_0\left(T_{MA}(t)-T_{MA}(t_s)\right)}\right]$$

Multiplying and dividing the sum by (S-1)! and taking the denominator outside the sum:

$$P=\left(\sum_{i=0}^{S-1}\sum_{j=0}^{S-1-i}\frac{e^{-\gamma_0 T_{MA}(t)}\left(\gamma_0 T_{MA}(t)\right)^i}{i!}\rho'^{\,i}_n\frac{e^{-\gamma_0 T_{FA}(t)}\left(\gamma_0 T_{FA}(t)\right)^j}{j!}\right)$$
$$+\int_0^t dt_s\left[\left(\rho'_r\gamma_{MA}(t_s)+\gamma_{FA}(t_s)\right)e^{-\gamma_0\left(T_{MA}(t_s)+T_{FA}(t_s)\right)}\right. \tag{B.7}$$
$$\left.\times\frac{1}{(S-1)!}\sum_{i=0}^{S-1}\left(\frac{(S-1)!}{i!(S-1-i)!}\left(\rho'_n\gamma_0 T_{MA}(t_s)\right)^i\left(\gamma_0 T_{FA}(t_s)\right)^{S-1-i}\right)e^{(\rho_0-1)\gamma_0\left(T_{MA}(t)-T_{MA}(t_s)\right)}\right]$$

The summation in line 2 can be recognised as a binomial series:

$$P=\left(\sum_{i=0}^{S-1}\sum_{j=0}^{S-1-i}\frac{e^{-\gamma_0 T_{MA}(t)}\left(\gamma_0 T_{MA}(t)\right)^i}{i!}\rho'^{\,i}_n\frac{e^{-\gamma_0 T_{FA}(t)}\left(\gamma_0 T_{FA}(t)\right)^j}{j!}\right)$$
$$+\int_0^t dt_s\left[\left(\rho'_r\gamma_{MA}(t_s)+\gamma_{FA}(t_s)\right)e^{-\gamma_0\left(T_{MA}(t_s)+T_{FA}(t_s)\right)}\frac{1}{(S-1)!}\left(\rho'_n\gamma_0 T_{MA}(t_s)+\gamma_0 T_{FA}(t_s)\right)^{S-1}\right. \tag{B.8}$$
$$\left.\times e^{(\rho_0-1)\gamma_0\left(T_{MA}(t)-T_{MA}(t_s)\right)}\right]$$

We the collect the exponential terms in line 2 and 3, and withdraws all factors not dependent upon t_s from the integration. We then have:

$$P=\left(\sum_{i=0}^{S-1}\sum_{j=0}^{S-1-i}\frac{e^{-\gamma_0 T_{MA}(t)}\left(\gamma_0 T_{MA}(t)\right)^i}{i!}\rho'^{\,i}_n\frac{e^{-\gamma_0 T_{FA}(t)}\left(\gamma_0 T_{FA}(t)\right)^j}{j!}\right)$$
$$+\frac{\gamma_0^{S-1}e^{(\rho_0-1)\gamma_0 T_{MA}(t)}}{(S-1)!}\int_0^t dt_s\left[e^{-\gamma_0\left(T_{FA}(t_s)+\rho_0 T_{MA}(t_s)\right)}\left(\rho'_r\gamma_{MA}(t_s)+\gamma_{FA}(t_s)\right)\left(\rho'_n T_{MA}(t_s)+T_{FA}(t_s)\right)^{S-1}\right] \tag{B.9}$$

To simplify the double sum, we can collect terms with common exponents of *i* and *j* and remove some factors outside the double sum:

$$P = e^{-\gamma_0(T_{MA}(t)+T_{FA}(t))}\left(\sum_{i=0}^{S-1}\sum_{j=0}^{S-1-i}\frac{\left(\rho'_n\,\gamma_0 T_{MA}(t)\right)^i}{i!}\frac{\left(\gamma_0 T_{FA}(t)\right)^j}{j!}\right)$$
$$+\frac{\gamma_0{}^{S-1}e^{(\rho_0-1)\gamma_0 T_{MA}(t)}}{(S-1)!}\int_0^t dt_s\left[e^{-\gamma_0(T_{FA}(t_s)+\rho_0 T_{MA}(t_s))}\left(\rho'_r\,\gamma_{MA}(t_s)+\gamma_{FA}(t_s)\right)\left(\rho'_n\,T_{MA}(t_s)+T_{FA}(t_s)\right)^{S-1}\right] \quad \text{(B.10)}$$

We now introduce new variables $u=i+j$ and $v=j$ and reorganise the terms in the double sum. By letting u vary between 0 and S-1 included, and v vary from 0 to u included, we account for all terms. We then have:

$$P = e^{-\gamma_0(T_{MA}(t)+T_{FA}(t))}\left(\sum_{u=0}^{S-1}\sum_{v=0}^{u}\frac{\left(\rho'_n\,\gamma_0 T_{MA}(t)\right)^{(u-v)}}{(u-v)!}\frac{\left(\gamma_0 T_{FA}(t)\right)^v}{v!}\right)$$
$$+\frac{\gamma_0{}^{S-1}e^{(\rho_0-1)\gamma_0 T_{MA}(t)}}{(S-1)!}\int_0^t dt_s\left[e^{-\gamma_0(T_{FA}(t_s)+\rho_0 T_{MA}(t_s))}\left(\rho'_r\,\gamma_{MA}(t_s)+\gamma_{FA}(t_s)\right)\left(\rho'_n\,T_{MA}(t_s)+T_{FA}(t_s)\right)^{S-1}\right] \quad \text{(B.11)}$$

Inserting the unit ratio $u!/u!$ in the sum, we obtain:

$$P = e^{-\gamma_0(T_{MA}(t)+T_{FA}(t))}\left(\sum_{u=0}^{S-1}\frac{1}{u!}\sum_{v=0}^{u}\frac{u!}{(u-v)!v!}\left(\gamma_0 T_{FA}(t)\right)^v\left(\rho'_n\,\gamma_0 T_{MA}(t)\right)^{(u-v)}\right)$$
$$+\frac{\gamma_0{}^{S-1}e^{(\rho_0-1)\gamma_0 T_{MA}(t)}}{(S-1)!}\int_0^t dt_s\left[e^{-\gamma_0(T_{FA}(t_s)+\rho_0 T_{MA}(t_s))}\left(\rho'_r\,\gamma_{MA}(t_s)+\gamma_{FA}(t_s)\right)\left(\rho'_n\,T_{MA}(t_s)+T_{FA}(t_s)\right)^{S-1}\right] \quad \text{(B.12)}$$

The inner sum can be recognised as a binomial series, and we finally get:

$$P = e^{-\gamma_0(T_{MA}(t)+T_{FA}(t))}\left(\sum_{u=0}^{S-1}\frac{\left(\gamma_0 T_{FA}(t)+\rho'_n\,\gamma_0 T_{MA}(t)\right)^u}{u!}\right)$$
$$+\frac{\gamma_0{}^{S-1}e^{(\rho_0-1)\gamma_0 T_{MA}(t)}}{(S-1)!}\int_0^t dt_s\left[e^{-\gamma_0(T_{FA}(t_s)+\rho_0 T_{MA}(t_s))}\left(\rho'_r\,\gamma_{MA}(t_s)+\gamma_{FA}(t_s)\right)\left(\rho'_n\,T_{MA}(t_s)+T_{FA}(t_s)\right)^{S-1}\right] \quad \text{(B.13)}$$

Alternatively we can choose to replace the sum by an integral. We then multiply with a unit ratio:

$$P = e^{-\gamma_0(T_{MA}(t)+T_{FA}(t))}e^{(\gamma_0 T_{FA}(t)+\rho'_n\gamma_0 T_{MA}(t))}\left(\sum_{u=0}^{S-1}\frac{e^{-(\gamma_0 T_{FA}(t)+\rho'_n\gamma_0 T_{MA}(t))}\left(\gamma_0 T_{FA}(t)+\rho'_n\,\gamma_0 T_{MA}(t)\right)^u}{u!}\right)$$
$$+\frac{\gamma_0{}^{S-1}e^{(\rho_0-1)\gamma_0 T_{MA}(t)}}{(S-1)!}\int_0^t dt_s\left[e^{-\gamma_0(T_{FA}(t_s)+\rho_0 T_{MA}(t_s))}\left(\rho'_r\,\gamma_{MA}(t_s)+\gamma_{FA}(t_s)\right)\left(\rho'_n\,T_{MA}(t_s)+T_{FA}(t_s)\right)^{S-1}\right] \quad \text{(B.14)}$$

Inserting the integral form of the sum, we get:

$$P = e^{(\rho'_n - 1)\gamma_0 T_{MA}(t)} \times \left(1 - \frac{1}{(S-1)!}\int_0^t dt_s e^{-(\gamma_0 T_{FA}(t_s) + \rho'_n \gamma_0 T_{MA}(t_s))} \left(\gamma_{FA}(t_s) + \rho'_n \gamma_{MA}(t_s)\right)\left(\gamma_0 T_{FA}(t_s) + \rho'_n \gamma_0 T_{MA}(t_s)\right)^{S-1}\right) + \frac{\gamma_0^{S-1} e^{(\rho_0 - 1)\gamma_0 T_{MA}(t)}}{(S-1)!}\int_0^t dt_s \left[e^{-\gamma_0(T_{FA}(t_s) + \rho_0 T_{MA}(t_s))} \left(\rho'_r \gamma_{MA}(t_s) + \gamma_{FA}(t_s)\right)\left(\rho'_n T_{MA}(t_s) + T_{FA}(t_s)\right)^{S-1}\right] \tag{B.15}$$

We finally add together the two integrals:

$$P = \left(e^{(\rho'_n - 1)\gamma_0 T_{MA}(t)}\right) + \frac{\gamma_0^{S-1} e^{-\gamma_0 T_{MA}(t)}}{(S-1)!}\int_0^t \Big[dt_s \left(\rho'_n T_{MA}(t_s) + T_{FA}(t_s)\right)^{S-1} e^{-\gamma_0 T_{FA}(t_s)} \times \left(e^{\rho_0 \gamma_0 (T_{MA}(t) - T_{MA}(t_s))}\left(\rho'_r \gamma_{MA}(t_s) + \gamma_{FA}(t_s)\right) - e^{\rho'_n \gamma_0 (T_{MA}(t) - T_{MA}(t_s))}\left(\rho'_n \gamma_{MA}(t_s) + \gamma_{FA}(t_s)\right)\right)\Big] \tag{B.16}$$

C DEVELOPMENT OF THE GENERAL MULTI-MISSILE EXPRESSION FOUND IN CHAPTER 3.3

We start with equation (3.31) in which the Poisson distribution has been inserted:

$$P = \left(\sum_{k_1=0}^{S-1}\sum_{k_2=0}^{S-1-k_1} \cdot\cdot \sum_{k_b=0}^{S-1-\sum_{d=1}^{b-1} k_d} \cdot\cdot \sum_{k_B=0}^{S-1-\sum_{f=1}^{B-1} k_f} \sum_{j=0}^{S-1-\sum_{g=1}^{B} k_g} \left(\frac{e^{-\gamma_0 T_{FA}(t)}\left(\gamma_0 T_{FA}(t)\right)^j}{j!}\prod_{h=1}^{B}\frac{e^{-\gamma_0 T_{MA,h}(t)}\left(\gamma_0 T_{MA,h}(t)\right)^{k_h}}{k_h!}\rho'^{\,k_h}_{h,n}\right)\right.$$
$$+\int_0^t dt_s \left(\left[\gamma_{FA}(t_s) + \sum_{m=1}^{B}\left(\gamma_{MA,m}(t_s)\rho'_{m,r}\right)\right]\times\right.$$
$$\left[\sum_{k_1=0}^{S-1}\sum_{k_2=0}^{S-1-k_1} \cdots \sum_{k_b=0}^{S-1-\sum_{d=1}^{b-1} k_d} \cdots \sum_{k_B=0}^{S-1-\sum_{f=1}^{B-1} k_f}\left(\frac{e^{-\gamma_0 T_{FA}(t_S)}\left(\gamma_0 T_{FA}(t_S)\right)^{S-1-\sum_{g=1}^{B} k_g}}{\left(S-1-\sum_{g=1}^{B} k_g\right)!}\prod_{h=1}^{B}\frac{e^{-\gamma_0 T_{MA,h}(t_S)}\left(\gamma_0 T_{MA,h}(t_S)\right)^{k_h}}{k_h!}\rho'^{\,k_h}_{h,n}\right)\right]\times$$
$$\left.\left.\left[\prod_{h=1}^{B}\sum_{k_h=0}^{\infty}\frac{e^{-\gamma_0(T_{MA,h}(t) - T_{MA,h}(t_s))}\left(\gamma_0\left(T_{MA,h}(t) - T_{MA,h}(t_s)\right)\right)^{k_h}}{k_h!}\rho_{h,0}^{\,k_h}\right]\right)\right)$$

As in appendix B, we multiply the term in the last line with a unit exponential:

$$P=\left(\sum_{k_1=0}^{S-1}\sum_{k_2=0}^{S-1-k_1}..\sum_{k_b=0}^{S-1-\sum_{d=1}^{b-1}k_d}..\sum_{k_B=0}^{S-1-\sum_{f=1}^{B-1}k_f}\sum_{j=0}^{S-1-\sum_{g=1}^{B}k_g}\left(\frac{e^{-\gamma_0 T_{FA}(t)}\left(\gamma_0 T_{FA}(t)\right)^j}{j!}\prod_{h=1}^{B}\frac{e^{-\gamma_0 T_{MA,h}(t)}\left(\gamma_0 T_{MA,h}(t)\right)^{k_h}}{k_h!}\rho'_{h,n}{}^{k_h}\right)\right)$$

$$+\int_0^t dt_s\left(\left[\gamma_{FA}(t_s)+\sum_{m=1}^{B}\left(\gamma_{MA,m}(t_s)\rho'_{m,r}\right)\right]\times\right.$$

$$\left[\sum_{k_1=0}^{S-1}\sum_{k_2=0}^{S-1-k_1}....\sum_{k_b=0}^{S-1-\sum_{d=1}^{b-1}k_d}...\sum_{k_B=0}^{S-1-\sum_{f=1}^{B-1}k_f}\left(\frac{e^{-\gamma_0 T_{FA}(t_S)}\left(\gamma_0 T_{FA}(t_S)\right)^{S-1-\sum_{g=1}^{B}k_g}}{\left(S-1-\sum_{g=1}^{B}k_g\right)!}\prod_{h=1}^{B}\frac{e^{-\gamma_0 T_{MA,h}(t_S)}\left(\gamma_0 T_{MA,h}(t_S)\right)^{k_h}}{k_h!}\rho'_{h,n}{}^{k_h}\right)\right]\times \qquad \text{(C.1)}$$

$$\left.\left[\prod_{h=1}^{B}e^{(\rho_{h,0}-1)\gamma_0\left(T_{MA,h}(t)-T_{MA,h}(t_s)\right)}\sum_{k_h=0}^{\infty}\frac{e^{-\rho_{h,0}\gamma_0\left(T_{MA,h}(t)-T_{MA,h}(t_s)\right)}\left(\rho_{h,0}\gamma_0\left(T_{MA,h}(t)-T_{MA,h}(t_s)\right)\right)^{k_h}}{k_h!}\right]\right)$$

The last sum can be recognised as a sum of all the terms in a Poisson distribution, the sum then equalling 1. It is also possible to simplify the exponential term, which we move up to line 2:

$$P=\left(\sum_{k_1=0}^{S-1}\sum_{k_2=0}^{S-1-k_1}..\sum_{k_b=0}^{S-1-\sum_{d=1}^{b-1}k_d}..\sum_{k_B=0}^{S-1-\sum_{f=1}^{B-1}k_f}\sum_{j=0}^{S-1-\sum_{g=1}^{B}k_g}\left(\frac{e^{-\gamma_0 T_{FA}(t)}\left(\gamma_0 T_{FA}(t)\right)^j}{j!}\prod_{h=1}^{B}\frac{e^{-\gamma_0 T_{MA,h}(t)}\left(\gamma_0 T_{MA,h}(t)\right)^{k_h}}{k_h!}\rho'_{h,n}{}^{k_h}\right)\right)$$

$$+\int_0^t dt_s\left(\left[\gamma_{FA}(t_s)+\sum_{m=1}^{B}\left(\gamma_{MA,m}(t_s)\rho'_{m,r}\right)\right]\times\left[e^{\gamma_0\sum_{h=1}^{B}(\rho_{h,0}-1)\left(T_{MA,h}(t)-T_{MA,h}(t_s)\right)}\right]\times\right. \qquad \text{(C.2)}$$

$$\left.\left[\sum_{k_1=0}^{S-1}\sum_{k_2=0}^{S-1-k_1}...\sum_{k_b=0}^{S-1-\sum_{d=1}^{b-1}k_d}...\sum_{k_B=0}^{S-1-\sum_{f=1}^{B-1}k_f}\left(\frac{e^{-\gamma_0 T_{FA}(t_S)}\left(\gamma_0 T_{FA}(t_S)\right)^{S-1-\sum_{g=1}^{B}k_g}}{\left(S-1-\sum_{g=1}^{B}k_g\right)!}\prod_{h=1}^{B}\frac{e^{-\gamma_0 T_{MA,h}(t_S)}\left(\gamma_0 T_{MA,h}(t_S)\right)^{k_h}}{k_h!}\rho'_{h,n}{}^{k_h}\right)\right]\right)$$

We now take the exponentials out of the sums:

$$P = e^{-\gamma_0\left(T_{FA}(t)+\sum_{h=1}^{B} T_{MA,h}(t)\right)} \left(\sum_{k_1=0}^{S-1} \sum_{k_2=0}^{S-1-k_1} .. \sum_{k_b=0}^{S-1-\sum_{d=1}^{b-1} k_d} .. \sum_{k_B=0}^{S-1-\sum_{f=1}^{B-1} k_f} \sum_{j=0}^{S-1-\sum_{g=1}^{B} k_g} \left(\frac{\left(\gamma_0 T_{FA}(t)\right)^j}{j!} \prod_{h=1}^{B} \frac{\left(\rho'_{h,n}\,\gamma_0 T_{MA,h}(t)\right)^{k_h}}{k_h\,!} \right) \right.$$
$$+ e^{\gamma_0 \sum_{h=1}^{B}(\rho_{h,0}-1) T_{MA,h}(t)} \gamma_0^{S-1} \int_0^t dt_s \left(\left[\gamma_{FA}(t_s) + \sum_{m=1}^{B} \left(\gamma_{MA,m}(t_s)\,\rho'_{m,r} \right) \right] \times \left[e^{-\gamma_0\left(T_{FA}(t_S)+\sum_{h=1}^{B}\rho_{h,0} T_{MA,h}(t_s)\right)} \right] \times \right. \qquad \text{(C.3)}$$
$$\left. \left. \left[\sum_{k_1=0}^{S-1} \sum_{k_2=0}^{S-1-k_1} \cdots \sum_{k_b=0}^{S-1-\sum_{d=1}^{b-1} k_d} \cdots \sum_{k_B=0}^{S-1-\sum_{f=1}^{B-1} k_f} \left(\frac{T_{FA}(t_S)^{S-1-\sum_{g=1}^{B} k_g}}{\left(S-1-\sum_{g=1}^{B} k_g\right)!} \prod_{h=1}^{B} \frac{\left(\rho'_{h,n}\, T_{MA,h}(t_S)\right)^{k_h}}{k_h\,!} \right) \right] \right) \right)$$

We note the similarity in the multiple sums in line 1 and 3. To simplify this expression further, we want to identify multinomial series. Multinomials are generalisations of binomials, and we have:

$$\left(A_1 + A_2 + \ldots + A_N\right)^K = K! \sum_{k_1=0}^{K} \sum_{k_2=0}^{K-k_1} \sum_{k_3=0}^{K-k_1-k_2} \cdots \sum_{k_{N-1}=0}^{K-\sum_{g=1}^{N-2} k_g} \frac{A_N^{\,K-\sum_{c=1}^{N-1} k_c}}{\left(K-\sum_{c=1}^{N-1} k_c\right)!} \prod_{b=1}^{N-1} \frac{A_b^{\,k_b}}{k_b\,!} \qquad \text{(C.4)}$$

In line 3 of equation C.3 we can, by replacing *K* by *S*-1 and *N* by *B*+1 in (C.4), identify the multiple sums as a multinomial divided by a factorial, and we have:

$$P = e^{-\gamma_0\left(T_{FA}(t)+\sum_{h=1}^{B} T_{MA,h}(t)\right)} \left(\sum_{k_1=0}^{S-1} \sum_{k_2=0}^{S-1-k_1} .. \sum_{k_b=0}^{S-1-\sum_{d=1}^{b-1} k_d} .. \sum_{k_B=0}^{S-1-\sum_{f=1}^{B-1} k_f} \sum_{j=0}^{S-1-\sum_{g=1}^{B} k_g} \left(\frac{\left(\gamma_0 T_{FA}(t)\right)^j}{j!} \prod_{h=1}^{B} \frac{\left(\rho'_{h,n}\,\gamma_0 T_{MA,h}(t)\right)^{k_h}}{k_h\,!} \right) \right.$$
$$+ e^{\gamma_0 \sum_{h=1}^{B}(\rho_{h,0}-1) T_{MA,h}(t)} \gamma_0^{S-1} \int_0^t dt_s \left(\left[\gamma_{FA}(t_s) + \sum_{m=1}^{B} \left(\gamma_{MA,m}(t_s)\,\rho'_{m,r} \right) \right] \times \left[e^{-\gamma_0\left(T_{FA}(t_S)+\sum_{h=1}^{B}\rho_{h,0} T_{MA,h}(t_s)\right)} \right] \times \right. \qquad \text{(C.5)}$$
$$\left. \left. \left[\frac{1}{(S-1)!} \left(T_{FA}(t_S) + \sum_{g=1}^{B} \left(\rho'_{g,n}\, T_{MA,g}(t_S) \right) \right)^{S-1} \right] \right) \right)$$

Using (C.4), we have the following relation:

$$\sum_{K=0}^{M}\frac{\left(A_1+A_2+\ldots+A_N\right)^K}{K!}=\sum_{K=0}^{M}\left(\sum_{k_1=0}^{K}\sum_{k_2=0}^{K-k_1}\sum_{k_3=0}^{K-k_1-k_2}\cdots\sum_{k_{N-1}=0}^{K-\sum_{g=1}^{N-2}k_g}\frac{A_N^{\,K-\sum_{c=1}^{N-1}k_c}}{\left(K-\sum_{c=1}^{N-1}k_c\right)!}\prod_{b=1}^{N-1}\frac{A_b^{\,k_b}}{k_b!}\right)$$

$$=\sum_{k_N=0}^{M}\left(\sum_{k_1=0}^{M-k_N}\sum_{k_2=0}^{M-k_N-k_1}\sum_{k_3=0}^{M-k_N-k_1-k_2}\cdots\sum_{k_{N-1}=0}^{M-k_N-\sum_{g=1}^{N-2}k_g}\frac{A_N^{\,M-k_N-\sum_{c=1}^{N-1}k_c}}{\left(M-k_N-\sum_{c=1}^{N-1}k_c\right)!}\prod_{b=1}^{N-1}\frac{A_b^{\,k_b}}{k_b!}\right) \qquad \text{(C.6)}$$

$$=\sum_{k_N=0}^{M}\left(\sum_{k_1=0}^{M-k_N}\sum_{k_2=0}^{M-k_N-k_1}\sum_{k_3=0}^{M-k_N-k_1-k_2}\cdots\sum_{k_{N-1}=0}^{M-k_N-\sum_{g=1}^{N-2}k_g}\frac{A_N^{\,M-\sum_{c=1}^{N}k_c}}{\left(M-\sum_{c=1}^{N}k_c\right)!}\prod_{b=1}^{N-1}\frac{A_b^{\,k_b}}{k_b!}\right)$$

where we have made the variable change $K \to k_N = M - K$. We see that shifting the indices in the following way: $A_i \to A_{i+1}$, $A_N \to A_1$, and $k_i \to k_{i+1}$, $k_N \to k_1$, does not change the value of the expression in C.6, and we therefore have:

$$\sum_{K=0}^{M}\frac{\left(A_N+A_1+A_2+\ldots+A_{N-1}\right)^K}{K!}=\sum_{K=0}^{M}\frac{\left(A_1+A_2+\ldots+A_N\right)^K}{K!}$$

$$=\sum_{k_1=0}^{M}\sum_{k_2=0}^{M-k_1}\sum_{k_3=0}^{M-k_1-k_2}\sum_{k_4=0}^{M-k_1-k_2-k_3}\cdots\sum_{k_N=0}^{M-\sum_{g=1}^{N-1}k_g}\frac{A_N^{\,M-\sum_{c=1}^{N}k_c}}{\left(M-\sum_{c=1}^{N}k_c\right)!}\prod_{b=1}^{N-1}\frac{A_b^{\,k_b}}{k_b!} \qquad \text{(C.7)}$$

We make the variable change $k_N \to k'_N = \left(M-\sum_{g=1}^{N-1}k_g\right)-k_N$ in the last sum in line 3 of (C.7):

$$\sum_{K=0}^{M}\frac{\left(A_N+A_1+A_2+\ldots+A_{N-1}\right)^K}{K!}=\sum_{k_1=0}^{M}\sum_{k_2=0}^{M-k_1}\sum_{k_3=0}^{M-k_1-k_2}\sum_{k_4=0}^{M-k_1-k_2-k_3}\cdots\sum_{k'_N=0}^{M-\sum_{g=1}^{N-1}k_g}\frac{A_N^{\,k'_N}}{k'_N!}\prod_{b=1}^{N-1}\frac{A_b^{\,k_b}}{k_b!} \qquad \text{(C.8)}$$

We can now, by replacing *K* by *S*-1 and *N* by *B*+1 in (C.8) identify the multiple sums in line 1 of equation (C.5) as a version of the multiple sums in (C.8), and we can then rewrite (C.5) as:

$$P = e^{-\gamma_0\left(T_{FA}(t)+\sum_{h=1}^{B}T_{MA,h}(t)\right)}\left(\sum_{K=0}^{S-1}\frac{\left(\gamma_0 T_{FA}(t)+\sum_{h=1}^{B}\rho'_{h,n}\,\gamma_0 T_{MA,h}(t)\right)^K}{K!}\right.$$

$$\left. + e^{\gamma_0\sum_{h=1}^{B}(\rho_{h,0}-1)T_{MA,h}(t)}\gamma_0^{\,S-1}\int_0^t dt_s\left(\left[\gamma_{FA}(t_s)+\sum_{m=1}^{B}\left(\gamma_{MA,m}(t_s)\,\rho'_{m,r}\right)\right]\times\left[e^{-\gamma_0\left(T_{FA}(t_S)+\sum_{h=1}^{B}\rho_{h,0}T_{MA,h}(t_s)\right)}\right]\times\left[\frac{1}{(S-1)!}\left(T_{FA}(t_S)+\sum_{g=1}^{B}\left(\rho'_{g,n}\,T_{MA,g}(t_S)\right)\right)^{S-1}\right]\right)\right) \tag{C.9}$$

We can then take the factors not dependent on t_S out of the integral to finally obtain:

$$P = e^{-\gamma_0\left(T_{FA}(t)+\sum_{h=1}^{B}T_{MA,h}(t)\right)}\left(\sum_{K=0}^{S-1}\frac{\left(\gamma_0 T_{FA}(t)+\sum_{h=1}^{B}\rho'_{h,n}\,\gamma_0 T_{MA,h}(t)\right)^K}{K!}\right)+\frac{\gamma_0^{\,S-1}e^{\gamma_0\sum_{h=1}^{B}(\rho_{h,0}-1)T_{MA,h}(t)}}{(S-1)!}\times$$

$$\int_0^t dt_s\left[\gamma_{FA}(t_s)+\sum_{m=1}^{B}\left(\gamma_{MA,m}(t_s)\,\rho'_{m,r}\right)\right]\left[e^{-\gamma_0\left(T_{FA}(t_S)+\sum_{h=1}^{B}\rho_{h,0}T_{MA,h}(t_s)\right)}\right]\left(T_{FA}(t_S)+\sum_{g=1}^{B}\left(\rho'_{g,n}\,T_{MA,g}(t_S)\right)\right)^{S-1} \tag{C.10}$$

D DEVELOPMENT OF UPPER LIMITS OF THE MISSION SURVIVAL PROBABILITY

The first upper limit to be found is $e^{(\rho'_n-1)\gamma_{MA}t}$. We must then determine the difference between this function and P to be greater than, or equal to 0. We us the integral definition of P derived in chapter 3.1 to get:

$$\begin{aligned}
& e^{(\rho'_n-1)\gamma_{MA}t} - P \\
&= e^{(\rho'_n-1)\gamma_{MA}t} - e^{(\rho'_n-1)\gamma_{MA}t} - \frac{\left(\rho'_n\,\gamma_{MA}+\gamma_{FA}\right)^S}{(S-1)!}\int_0^t dt_s t_s^{\,S-1}\left(e^{(\rho_0-1)\gamma_{MA}t-(\rho_0\gamma_{MA}+\gamma_{FA})t_s} - e^{(\rho'_n-1)\gamma_{MA}t-(\rho'_n\gamma_{MA}+\gamma_{FA})t_s}\right) \\
&= -\frac{\left(\rho'_n\,\gamma_{MA}+\gamma_{FA}\right)^S}{(S-1)!}\int_0^t\left(dt_s t_s^{\,S-1}e^{-\gamma_{MA}t-\gamma_{FA}t_s}\left(e^{\rho_0\gamma_{MA}(t-t_s)} - e^{\rho'_n\gamma_{MA}(t-t_s)}\right)\right) \\
&\geq 0
\end{aligned} \tag{D.1}$$

The expression $e^{(\rho'_n-1)\gamma_{MA}t}$ is therefore an upper limit for the mission survival probability.

We now develop another upper limit, namely $\left(\dfrac{\gamma_{FA}+\rho'_n\,\gamma_{MA}}{\gamma_{FA}+\rho_0\gamma_{MA}}\right)^S e^{(\rho_0-1)\gamma_{MA}t}$. We have:

$$
\begin{aligned}
&\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^S e^{(\rho_0-1)\gamma_{MA}t}-P\\
&=\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^S e^{(\rho_0-1)\gamma_{MA}t}-\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^S e^{(\rho_0-1)\gamma_{MA}t}\\
&\quad-\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^S e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{u-S}-\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S}\right)\right)\\
&=\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^S e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S}-\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{u-S}\right)\right)\\
&\geq 0
\end{aligned}
\tag{D.2}
$$

The expression $\left(\frac{\gamma_{FA}+\rho'_n\gamma_{MA}}{\gamma_{FA}+\rho_0\gamma_{MA}}\right)^S e^{(\rho_0-1)\gamma_{MA}t}$ **is therefore an upper limit for the mission survival probability.**

E DERIVATION OF THE PARTIAL DERIVATIVE OF THE MISSION SURVIVAL PROBABILITY WITH RESPECT TO THE FALSE ALARM RATE

We start by taking the partial derivative of the summation form of the mission survival probability:

$$
\begin{aligned}
\frac{\partial P}{\partial\gamma_{FA}}&=\frac{\partial}{\partial\gamma_{FA}}\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^S e^{(\rho_0-1)\gamma_{MA}t}\\
&\quad+\frac{\partial}{\partial\gamma_{FA}}\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^S e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{u-S}-\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S}\right)\right)\right)\\
&=\frac{S\gamma_{MA}\left(\rho_0-\rho'_n\right)}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^2}\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^{S-1}e^{(\rho_0-1)\gamma_{MA}t}\\
&\quad+\frac{\partial}{\partial\gamma_{FA}}\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{u}-\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{S}\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S}\right)\right)\right)\\
&=\frac{S\gamma_{MA}\left(\rho_0-\rho'_n\right)}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^2}\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^{S-1}e^{(\rho_0-1)\gamma_{MA}t}\\
&\quad-t\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{u}-\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{S}\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S}\right)\right)\right)\\
&\quad+\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(u\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{u-1}\right)\right)\right)\\
&\quad-\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(S\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{S-1}\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S}\right)\right)\right)\\
&\quad-\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{S}\left(u-S\right)\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S-1}\right)\right)\right)
\end{aligned}
\tag{E.1}
$$

We split the factor (*u*-*S*) in the last sum, and remove all the summations with *u* in the numerator for *u*=0:

$$
\begin{aligned}
&\frac{\partial P}{\partial \gamma_{FA}} = \frac{S\gamma_{MA}\left(\rho_0 - \rho'_n\right)}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^2}\left(\frac{\rho'_n\,\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^{S-1} e^{(\rho_0-1)\gamma_{MA}t} \\
&-t\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\,\gamma_{MA}+\gamma_{FA}\right)^u - \left(\rho'_n\,\gamma_{MA}+\gamma_{FA}\right)^S\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S}\right)\right)\right) \\
&+\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=1}^{S-1}\frac{t^u}{u!}\left(u\left(\rho'_n\,\gamma_{MA}+\gamma_{FA}\right)^{u-1}\right)\right)\right) \\
&-\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(S\left(\rho'_n\,\gamma_{MA}+\gamma_{FA}\right)^{S-1}\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S}\right)\right)\right) \\
&-\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=1}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\,\gamma_{MA}+\gamma_{FA}\right)^S u\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S-1}\right)\right)\right) \\
&+\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\,\gamma_{MA}+\gamma_{FA}\right)^S S\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S-1}\right)\right)\right)
\end{aligned}
\tag{E.2}
$$

We now simplify the terms *u/u!*, take *t* outside some of the sums, and reorder the terms:

$$
\begin{aligned}
\frac{\partial P}{\partial \gamma_{FA}} = &\frac{S\gamma_{MA}\left(\rho_0 - \rho'_n\right)}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^2}\left(\frac{\rho'_n\,\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^{S-1} e^{(\rho_0-1)\gamma_{MA}t} \\
&-t\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\,\gamma_{MA}+\gamma_{FA}\right)^u - \left(\rho'_n\,\gamma_{MA}+\gamma_{FA}\right)^S\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S}\right)\right)\right) \\
&+t\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=1}^{S-1}\frac{t^{u-1}}{(u-1)!}\left(\left(\rho'_n\,\gamma_{MA}+\gamma_{FA}\right)^{u-1}\right)\right)\right) \\
&-t\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=1}^{S-1}\frac{t^{u-1}}{(u-1)!}\left(\left(\rho'_n\,\gamma_{MA}+\gamma_{FA}\right)^S\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-1-S}\right)\right)\right) \\
&+S\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\,\gamma_{MA}+\gamma_{FA}\right)^S\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S-1}\right)\right)\right) \\
&-S\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\,\gamma_{MA}+\gamma_{FA}\right)^{S-1}\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S}\right)\right)\right)
\end{aligned}
\tag{E.3}
$$

We now make a variable change in line 3 (u→u+1), change the limits correspondingly, and regroup common terms in line 4:

$$\frac{\partial P}{\partial \gamma_{FA}} = \frac{S\gamma_{MA}\left(\rho_0 - \rho'_n\right)}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^2}\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^{S-1} e^{(\rho_0-1)\gamma_{MA}t}$$
$$-t\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^u-\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^S\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S}\right)\right)\right)$$
$$+t\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-2}\frac{t^u}{u!}\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^u\right)\right)\right)-t\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-2}\frac{t^u}{u!}\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^S\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^u\right.\right.\right. \quad \text{(E.4)}$$
$$+S\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\rho'_n\gamma_{MA}+\gamma_{FA}-\rho_0\gamma_{MA}-\gamma_{FA}\right)\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{S-1}\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S-1}\right)\right)\right)$$

We can now regroup the terms of line 2 and 3, and reorganise the last line:

$$\frac{\partial P}{\partial \gamma_{FA}} = \frac{S\gamma_{MA}\left(\rho_0 - \rho'_n\right)}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^2}\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^{S-1} e^{(\rho_0-1)\gamma_{MA}t}$$
$$-t\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\frac{t^{S-1}}{(S-1)!}\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{S-1}-\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^S\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{-1}\right)\right) \quad \text{(E.5)}$$
$$+S\gamma\left(e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\rho'_n-\rho_0\right)\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\left(\rho'_n\gamma_{MA}+\gamma_{FA}\right)^{S-1}\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^{u-S-1}\right)\right)\right)$$

We then simplify line 2 and 3:

$$\frac{\partial P}{\partial \gamma_{FA}} = \frac{S\gamma\left(\rho_0 - \rho'_n\right)}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^2}\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^{S-1} e^{(\rho_0-1)\gamma_{MA}t}$$
$$-\frac{S\gamma_{MA}\left(\rho_0 - \rho'_n\right)}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^2}\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^{S-1} e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\frac{t^S}{S!}\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^S\right) \quad \text{(E.6)}$$
$$-\frac{S\gamma_{MA}\left(\rho_0 - \rho'_n\right)}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^2}\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^{S-1} e^{-(\gamma_{MA}+\gamma_{FA})t}\left(\sum_{u=0}^{S-1}\frac{t^u}{u!}\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^u\right)$$

The merger of line 2 and 3 leads to:

$$\frac{\partial P}{\partial \gamma_{FA}} = \frac{S\gamma_{MA}\left(\rho_0 - \rho'_n\right)}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^2}\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^{S-1} e^{(\rho_0-1)\gamma_{MA}t}$$
$$-\frac{S\gamma_{MA}\left(\rho_0 - \rho'_n\right)}{\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)}\left(\frac{\rho'_n\gamma_{MA}+\gamma_{FA}}{\rho_0\gamma_{MA}+\gamma_{FA}}\right)^{S-1} e^{(\rho_0-1)\gamma_{MA}t}\left(\sum_{u=0}^{S}\frac{e^{-(\rho_0\gamma_{MA}+\gamma_{FA})t}t^u}{u!}\left(\left(\rho_0\gamma_{MA}+\gamma_{FA}\right)^u\right)\right) \quad \text{(E.7)}$$

We finally merge the terms in line 1 and 2, to finally obtain:

$$\frac{\partial P}{\partial \gamma_{FA}} = \frac{S\gamma_{MA}\left(\rho_0 - \rho'_n\right)}{\left(\rho_0\gamma_{MA} + \gamma_{FA}\right)}\left(\frac{\rho'_n\,\gamma_{MA} + \gamma_{FA}}{\rho_0\gamma_{MA} + \gamma_{FA}}\right)^{S-1} e^{(\rho_0 - 1)\gamma_{MA}t}\left(1 - \sum_{u=0}^{S}\frac{e^{-(\rho_0\gamma_{MA} + \gamma_{FA})t}t^u}{u!}\left(\left(\rho_0\gamma_{MA} + \gamma_{FA}\right)^u\right)\right) \qquad \text{(E.8)}$$