p-ADIC BROWNIAN MOTION AS A LIMIT OF DISCRETE TIME RANDOM WALKS

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Abstract. The p-adic diffusion equation is a pseudo differential equation that is formally analogous to the real diffusion equation. The fundamental solutions to pseudo differential equations that generalize the p-adic diffusion equation give rise to p-adic Brownian motions. We show that these stochastic processes are similar to real Brownian motion in that they arise as limits of discrete time random walks on grids. While similar to those in the real case, the random walks in the p-adic setting are necessarily non-local. The study of discrete time random walks that converge to Brownian motion provides intuition about Brownian motion that is important in applications and such intuition is now available in a non-Archimedean setting.

Contents

1. Introduction

The study of the convergence of discrete time random walks on scaled integer lattices to Brownian motion is both classical and foundational to the subject of probability. For any given positive real number D, the diffusion equation in the real setting with diffusion constant D is the partial differential equation

$$\frac{\partial \phi}{\partial t}(t,x) = \frac{D}{2}\frac{\partial^2 \phi}{\partial x^2}(t,x). \tag{1}$$

The fundamental solution, ρ, of (1) is given by

$$\rho(t,x) = \frac{1}{\sqrt{2\pi D t}} \exp\left(-\frac{x^2}{2Dt}\right).$$

The function ρ is a probability density function on the set of real numbers, $\mathbb{R}$, and gives rise to a Wiener measure W, a probability measure on the set $C([0,\infty)\colon \mathbb{R})$ of continuous functions defined on $[0,\infty)$ and valued in $\mathbb{R}$ that gives full measure to the set of paths that are initially at zero at time zero. The stochastic process X that maps each non-negative real t to the random variable X_t, where X_t acts on the probability space $(C([0,\infty)\colon \mathbb{R}), W)$ by

$$X_t(\omega) = \omega(t) \quad \text{with} \quad \omega \in C([0,\infty)\colon \mathbb{R}),$$

is a Brownian motion. For each positive t, the random variable X_t is mean free with variance Dt. The space of continuous paths is a closed subspace of the Skorohod space $D([0,\infty)\colon \mathbb{R})$, the set of càdlàg functions mapping $[0,\infty)$ to $\mathbb{R}$ endowed with the Skorohod metric. A central result in the theory of convergence of stochastic processes is that Wiener measure is the weak-$*$ limit of probability measures on $D([0,\infty)\colon \mathbb{R})$ that are concentrated on the step functions and associated to a sequence of discrete time stochastic processes known as discrete time random walks. In this precise sense, Brownian motion is a limit of discrete time random walks [8], [9].

The study of p-adic random walks and of integration over p-adic path spaces has a long history and is ongoing. Vladimirov introduced in [19] a pseudo differential operator that is, in many respects, an analog of the classical Laplacian in the p-adic setting and further investigated the spectral properties of this operator in [20]. Vladimirov and Volovich together initiated the study of quantum systems in the p-adic setting with their seminal articles [21] and [22]. Ismagilov studied the spectra of self-adjoint operators in [15] in the setting of $L^2(K)$, where K is a local field. In [24], Zelenov studied Feynman integrals with p-adic valued paths. Kochubei gave not only the fundamental solution to the p-adic analog of the diffusion equation in [17], using the operator introduced by Vladimirov, but also developed a theory of p-adic diffusion equations and a Feynman-Kac formula for the operator semigroup with a p-adic Schrödinger type operator as its infinitesimal generator. Albeverio and Karwowski further investigated diffusion in the p-adic setting in [1], constructing a continuous time random walk on $\mathbb{Q}_p$, computing its transition semigroup and infinitesimal generator, and showing among other things that the associated Dirichlet form is of jump type.

We follow the approach of [18], in which Varadarajan discussed an analog to the diffusion equation in the non-Archimedean setting in the general context where the functions have domains contained in $[0,\infty)\times\mathcal{S}$, where $\mathcal{S}$ is a finite dimensional vector space over a division ring which is finite dimensional over a local field of arbitrary characteristic. Preferring accessibility over generality, we specialize to the case where $\mathcal{S}$ is the field of p-adic numbers, $\mathbb{Q}_p$. The results of [18] specialize to show that the fundamental solutions to certain pseudo-differential equations formally analogous to the diffusion equation in the p-adic setting give rise to measures on the Skorohod space $D([0,\infty)\colon \mathbb{Q}_p)$ of càdlàg functions defined on $[0,\infty)$ and valued in $\mathbb{Q}_p$. Given such a measure P on $D([0,\infty)\colon \mathbb{Q}_p)$, the associated stochastic process X that maps each non-negative real t to the random variable X_t, where X_t acts on $D([0,\infty)\colon \mathbb{Q}_p)$ by

$$X_t(\omega) = \omega(t) \quad \text{with} \quad \omega \in D([0,\infty)\colon \mathbb{Q}_p),$$

is a process with independent increments and a p-adic analog of a Brownian motion. The probability density function f_t for the random variable X_t is a solution to the pseudo-differential equation that gives rise to P. We show that the well-known convergence in the real setting of discrete time random walks to Brownian motion has an analog in the p-adic setting, demonstrating that the analogy between the p-adic diffusion equation and the real diffusion equation is the result of a general principal of convergence of discrete time random walks on grids to a continuum limit. Whereas earlier articles such as [6] and [7] discuss the convergence of sequences of continuous time random walks on grids in local fields to a continuum limit, this article differs in that it studies the convergence of discrete time random walks. Discrete time random walks approximating p-adic diffusion offer greater intuition about their continuum limit but are, as one should expect, more difficult to study than the continuous time approximations.

The last two decades have seen considerable interest in p-adic mathematical physics. The book [23] seems to be the first textbook on p-adic mathematical physics and is still a standard reference in the field. The more recent article [11] gives a comprehensive overview of p-adic mathematical physics, as of the year 2009, and a detailed list of references that document the development of the subject. While there is intrinsic interest in the study of non-Archimedean analogs of Brownian motion, these analogs are also of interest because of their potential application to the study of physical systems. Ultrametricity arises naturally in the theory of complex systems and the many references cited by [16, Chapter 4] study an area in which the present work should find direct application, for example, in the works [2, 3, 4, 5] of Avetisov, Bikulov, Kozyrev, and Osipov dealing with p-adic models for complex systems. There are also potential applications of discrete time p-adic random walks in the study of the fractal properties of p-adic spaces. For instance, [13] and [14] investigate p-adic fractal strings and their complex dimensions. The p-adic Brownian path spaces offer a new setting in which to study the theory of complex dimension and discrete time p-adic random walks already appear useful in developing intuition about dimension in this context. Discrete time random walks that converge to real Brownian motion give intuition about the properties of real Brownian motion and give insight into these more complicated processes. The current paper promises to bring similar intuition and insight to the setting of p-adic diffusion.

For the convenience of the reader and to clarify the exposition, the following two sections present background information necessary to the presentation and proof of the main result. The fourth section gives a roadmap of the proof of the main theorem of the paper and should make the paper more accessible. The fifth section defines a discrete time random walk on a discrete space that will serve as a primitive process. It goes on to define a family of spatiotemporal embeddings of this process that introduce time and distance scales. Each of these spatiotemporal embeddings associates a different measure to a single path space with paths valued in the p-adic numbers. For each positive real number T, denote by $D([0,T]\colon \mathbb{Q}_p)$ the Skorohod space of paths on $[0,T]$ valued in $\mathbb{Q}_p$. The final section proves that, given a probability measure P on $D([0,T]\colon \mathbb{Q}_p)$ associated to a p-adic Brownian motion, there is a sequence $\big(P^{(n)}\big)$ of measures on $D([0,T]\colon \mathbb{Q}_p)$ associated to spatiotemporal embeddings of the primitive stochastic process such that P is the weak-$*$ limit of the $P^{(n)}$.

2. Background and Notation

2.1. **Path Space Terminology.** Assume throughout this section that the set $\mathcal{S}$ is a Polish space.

Definition 1. A *continuous time interval* I is an interval in $\mathbb{R}$ with left endpoint included and equal to zero. A *discrete time interval* I is a discrete subset of $[0,\infty)$ that contains zero. A *time interval* is either a continuous or discrete time interval.

Definition 2. Let I be a time interval and denote by $F(I\colon \mathcal{S})$ the set of all functions ω with

$$\omega\colon I \to \mathcal{S}.$$

The set $F(I\colon \mathcal{S})$ is the *space of all paths in $\mathcal{S}$ with domain I.*

Definition 3. A set $\mathcal{P}$ is a *path space* of $\mathcal{S}$ if there is a time interval I such that $\mathcal{P}$ is a subset of $F(I\colon \mathcal{S})$.

Definition 4. An *epoch* for a path space $\mathcal{P}$ with time interval I is a strictly increasing finite sequence e that is valued in $I\setminus\{0\}$.

Definition 5. A set h is a *history* for a path space $\mathcal{P}$ with time interval I if there is a natural number k and an epoch e with

$$e = (t_1,\dots,t_k),$$

and Borel subsets $U_0,\dots,U_k$ of $\mathcal{S}$ such that

$$h = \left((0,U_0)\,,(t_1,U_1)\,,\dots,(t_k,U_k)\right).$$

The finite sequence U with

$$U = (U_0,U_1,\dots,U_k)$$

is said to be the *route* of h.

If h is a history, then denote by $e(h)$ the epoch associated to h, by $U(h)$ the route associated to h, and by $\ell(h)$ the number of places of $e(h)$, the *length* of h. The language established in the above definitions allows us to say that a history is the pairing of an epoch and an initial time point with a route with one more place than the number of places of the epoch, a starting location that is paired with the initial time.

Definition 6. Let H be the set of all histories for $\mathcal{P}$. Define a function C that associates to each h in H a set $C(h)$ by

$$C(h) = \{\omega \in \mathcal{P} \colon \omega(0) \in U(h)_0 \\ \text{and } i \in \mathbb{N} \cap (0, \ell(h)] \implies \omega\left(e(h)_i\right) \in U(h)_i\} .$$

The set $C(H)$ is said to be the set of all *cylinder sets* of $\mathcal{P}$ and a set is said to be a *cylinder set* if it is an element of $C(H)$.

Remark. Since the route may take the empty set as a value, the set $C(H)$ contains the empty set and forms an algebra.

Given suitable restrictions on a pre-measure on the set of cylinder sets of paths in $\mathcal{S}$ with epochs valued in a fixed time interval I, the Kolmogorov Extension Theorem permits an extension of the pre-measure to a measure on $F(I \colon \mathcal{S})$. Study of analytical questions regarding diffusion usually requires specialization to a subset of $F(I \colon \mathcal{S})$ with paths that have nicer analytical properties. Two path spaces are of particular importance in the study of diffusion, namely, the set $C(I \colon \mathcal{S})$ of continuous functions from I to $\mathcal{S}$ equipped with the topology of uniform convergence on compacta and the *Skorohod space* $D(I \colon \mathcal{S})$ of càdlàg functions from I to $\mathcal{S}$ equipped with the Skorohod metric.

2.2. **Basic Facts about $\mathbb{Q}_p$.** The following two subsections introduce the space of p-adic numbers and the p-adic diffusion equation. These subsections borrow heavily from [6] with only minor changes.

Fix a prime number, p, and denote by $\mathbb{Q}_p$ the field of p-adic numbers, the completion of the rational numbers with respect to the p-adic valuation $|\cdot|$. For each x in $\mathbb{Q}_p$ and each integer k, denote by $B_k(x)$ and $S_k(x)$ the sets

$$B_k(x) = \{y \in \mathbb{Q}_p \colon |y - x| \leq p^k\} \quad \text{and} \quad S_k(x) = \{y \in \mathbb{Q}_p \colon |y - x| = p^k\}.$$

Denote by $\mathbb{Z}_p$ the *ring of integers*, the set $B_0(0)$. Let μ be the Haar measure on the additive group $\mathbb{Q}_p$, normalized so that the measure of $\mathbb{Z}_p$ is equal to one. Uniquely associated to each x in $\mathbb{Q}_p$ is a function

$$a_x \colon \mathbb{Z} \to \{0, 1, \ldots, p-1\} \tag{2}$$

such that

$$x = \sum_{k \in \mathbb{Z}} a_x(k) p^k$$

and for some natural number N,

$$k > N \implies a_x(-k) = 0.$$

For each x in $\mathbb{Q}_p$, define $\{x\}$ by

$$\{x\} = \sum_{k<0} a_x(k) p^k.$$

The function

$$\chi\colon \mathbb{Q}_p \to S^1 \quad \text{by} \quad \chi(x) = \mathrm{e}^{2\pi i\{x\}}$$

is a rank zero character on $\mathbb{Q}_p$, this is to say that χ is identically equal to one on $\mathbb{Z}_p$ and is not identically one on any ball centered at the origin with radius larger than one. The locally compact group $\mathbb{Q}_p$ is self dual—for any character ϕ on $\mathbb{Q}_p$, there is an α in $\mathbb{Q}_p$ so that for all x in $\mathbb{Q}_p$,

$$\phi(x) = \chi(\alpha x).$$

Denote by $\mathcal{F}$ the Fourier transform on $L^2(\mathbb{Q}_p)$, the unitary extension to $L^2(\mathbb{Q}_p)$ of the operator initially defined as the unitary operator mapping $L^1(\mathbb{Q}_p) \cap L^2(\mathbb{Q}_p)$ to $L^2(\mathbb{Q}_p)$ by

$$(\mathcal{F}f)(x) = \int_{\mathbb{Q}_p} \chi(-xy) f(y)\,\mathrm{d}y.$$

Denote by $\mathcal{F}^{-1}$ the inverse Fourier transform. For all f in $L^1(\mathbb{Q}_p) \cap L^2(\mathbb{Q}_p)$,

$$(\mathcal{F}^{-1}f)(y) = \int_{\mathbb{Q}_p} \chi(xy) f(x)\,\mathrm{d}x.$$

Denote by $SB(\mathbb{Q}_p)$ the *Schwartz-Bruhat* space of complex-valued, compactly supported, locally constant functions on $\mathbb{Q}_p$. This set of functions is the p-adic analog of the set of complex-valued, compactly supported, smooth functions on $\mathbb{R}$ with the important difference that, unlike in the real case, $SB(\mathbb{Q}_p)$ is invariant under the Fourier transform.

2.3. **The Diffusion Equation in $\mathbb{Q}_p$.** Fix a positive real number b and define the multiplication operator M on $SB(\mathbb{Q}_p)$ by

$$(Mf)(x) = |x|^b f(x).$$

Denote by Δ_0 the *pseudo Laplace operator* that maps $SB(\mathbb{Q}_p)$ to $L^2(\mathbb{Q}_p)$ by

$$(\Delta_0 f)(x) = \big(\mathcal{F}^{-1} M \mathcal{F} f\big)(x).$$

This operator is essentially self-adjoint on $SB(\mathbb{Q}_p)$ (see [12]). The self-adjoint closure of Δ_0, denoted by Δ, is a densely defined, unbounded, self-adjoint operator on $L^2(\mathbb{Q}_p)$, the *Vladimirov operator* on $L^2(\mathbb{Q}_p)$ with domain $\mathcal{D}(\Delta)'$. Denote by $\mathcal{D}(\Delta)$ the set of complex valued function on $\mathbb{R}_+ \times \mathbb{Q}_p$ with the property that if f is in $\mathcal{D}(\Delta)$, and t is a positive real number, then the function $f(t,\cdot)$ is in $\mathcal{D}(\Delta)'$. View the Vladimirov operator as acting on functions f in $\mathcal{D}(\Delta)$ by

$$(\Delta f)(t,x) = (\Delta f(t,\cdot))(x).$$

Define the Fourier and inverse Fourier transforms as acting on functions on $\mathbb{R}_+ \times \mathbb{Q}_p$ that for each positive t are square integrable over $\mathbb{Q}_p$ by computing the given transform of the function for fixed positive t. Define the derivative with respect to t of a function f that acts on $\mathbb{R}_+ \times \mathbb{Q}_p$ by fixing the p-adic argument and viewing f as a function on $\mathbb{R}_+$. Fix D to be a positive real number. The pseudo differential equation

$$\frac{\mathrm{d}f(t,x)}{\mathrm{d}t} = -D\Delta f(t,x) \tag{3}$$

has as its fundamental solution the function

$$f(t,x) = \left(\mathcal{F}^{-1}\mathrm{e}^{-Dt|\cdot|^b}\right)(x).$$

The more general result of [18] specializes to show that $f(t,x)$ is a probability density function that gives rise to a probability measure P on $D([0,\infty)\colon \mathbb{Q}_p)$ that is concentrated on the set of paths that are at zero at time zero. Fix a particular history h. For each i in $(0,\ell(h)] \cap \mathbb{N}$, denote the time point t_i and the Borel set V_i by

$$t_i = e(h)_i \quad \text{and} \quad V_i = U(h)_i.$$

If zero is contained in $U(h)_0$, then define $P(C(h))$ by

$$P(C(h)) = \int_{V_1} \cdots \int_{V_n} f(t_1,x_1) \\ f(t_2 - t_1, x_2 - x_1) \cdots f(t_n - t_{n-1}, x_n - x_{n-1})\,\mathrm{d}x_n \cdots \mathrm{d}x_1.$$

If zero is not in $U(h)_0$, then $P(C(h))$ is zero. This pre-measure on the cylinder sets extends to a measure on $D([0,\infty)\colon \mathbb{Q}_p)$. On restricting time points to a compact continuous time interval I, the results of [18] specialize to show that the pre-measure P_I that arises from the same density function as P extends to a measure on $D(I\colon \mathbb{Q}_p)$.

3. A Roadmap of the Proof

The proof of convergence of discrete time random walks to a p-adic Brownian motion that the current paper presents closely follows the standard proof in the real setting. However, there are notable differences in where difficulties arise that merit discussion.

The real diffusion equation gives rise to a continuous time stochastic process whose components are random variables acting on a single space of paths on which there is a measure that determines all probabilities associated with the stochastic process. The same is true of the p-adic diffusion equation. In the real setting, this measure is a Wiener measure on the space of continuous paths, which form a closed subset of the Skorohod space, $D(I\colon \mathbb{R})$. In the p-adic case, the sample paths will not be continuous since the only continuous paths are the constant paths. The goal is to show that, in both settings, these measures on their respective path spaces are weak-$*$ limits of sequences of measures associated to discrete time random walks.

It is convenient and conceptually appealing to introduce a discrete time random walk abstractly, by defining the probabilities associated to the values that the various random variables of the process assume rather than by making assumptions about the underlying space on which the random variables act. General results about extensions of pre-measures on probability spaces permit the construction of a concrete model of the abstractly defined process, a single probability space on which all random variables specified by the process act and whose measure gives rise to the finite dimensional distributions specified by the abstractly defined process. This space will not be unique and authors frequently refer to different models that give rise to the same finite dimensional distributions as different versions of the process, a nomenclature the current paper adopts.

Rather than initially consider a sequence of random walks on embedded grids in $\mathbb{R}$ (in the real case) or in $\mathbb{Q}_p$ (in the p-adic case), take the perspective that in each case there is but one *primitive* random walk without any naturally associated fundamental time or length scale and that this random walk is simply a stochastic process mapping $\mathbb{N}_0$, the natural numbers with zero included, to random variables all defined on the same sample space. In the real setting, if m is in $\mathbb{N}_0$ and $\tilde{S}$ is the primitive random walk, then $\tilde{S}_m$ is valued in the integers. If $\tilde{U}$ is an abstractly defined random variable and A is a subset of a set in which $\tilde{U}$ takes values, then denote by $\operatorname{Prob}(\tilde{U} \in A)$ the probability of the event that $\tilde{U}$ assumes a value in A and if a is in A, define

$$\operatorname{Prob}\big(\tilde{U} = a\big) \coloneqq \operatorname{Prob}\big(\tilde{U} \in \{a\}\big).$$

Let $\tilde{X}$ be the abstractly defined random variable with distribution

$$\begin{cases} \operatorname{Prob}\big(\tilde{X} = -1\big) = \frac{1}{2} \\ \operatorname{Prob}\big(\tilde{X} = 1\big) = \frac{1}{2}. \end{cases}$$

Let $\big(\tilde{X}_i\big)_{i\in\mathbb{N}}$ be a sequence of independent identically distributed random variables, each with the same distribution as $\tilde{X}$. Let $\tilde{X}_0$ be equal to zero with probability one. For each m in $\mathbb{N}_0$, denote by $\tilde{S}_m$ the random variable

$$\tilde{S}_m = \tilde{X}_0 + \tilde{X}_1 + \cdots + \tilde{X}_m.$$

The random variable $\tilde{S}_m$ has mean equal to zero and has variance equal to m. The sequence of random variables $\tilde{S}$, with

$$\tilde{S} \colon m \mapsto \tilde{S}_m \quad (m \in \mathbb{N}_0),$$

is an abstractly defined $\mathbb{Z}$–valued stochastic process.

The probabilities given by the process $\tilde{S}$ define a pre-measure on the cylinder sets of $F(\mathbb{N}_0 \colon \mathbb{Z})$, taking a cylinder set $C(h)$ with history h to the value $P(C(h))$ defined in the following way. If for each i in $\{1, \dots, \ell(h)\}$,

$$t_i = e(h)_i, \quad U_i = U(h)_i, \quad \text{and} \quad \ell(h) = \ell,$$

then

$$P(C(h)) = \sum_{x_0\in U_0}\sum_{x_1\in U_1}\cdots\sum_{x_\ell\in U_\ell} \mathrm{Prob}\left(\tilde{S}_0 = x_0\right)\mathrm{Prob}\left(\tilde{S}_{t_1} = x_1 - x_0\right)$$
$$\cdot\, \mathrm{Prob}\left(\tilde{S}_{t_2} - \tilde{S}_{t_1} = x_2 - x_1\right)\cdots\mathrm{Prob}\left(\tilde{S}_{t_\ell} - \tilde{S}_{t_{\ell-1}} = x_\ell - x_{\ell-1}\right).$$

The Kolmogorov Extension theorem guarantees the existence of a measure, P, on $F(\mathbb{N}_0\colon \mathbb{Z})$ that restricts to this pre-measure on the cylinder sets. Define for each n the random variable S_m by

$$S_m(\omega) = \omega(m) \quad \text{where} \quad \omega \in F(\mathbb{N}_0\colon \mathbb{Z}).$$

The sample space $(F(\mathbb{N}_0\colon \mathbb{Z}), P)$ together with the sequence S of random variables (S_m) forms the stochastic process $(F(\mathbb{N}_0\colon \mathbb{Z}), P, S)$, a concrete model for the process $\tilde{S}$.

The natural action of $\mathbb{Z}$ on $\mathbb{R}$ defines a sequence of embeddings of $\mathbb{Z}$ into $\mathbb{R}$. In particular, if (δ_n) is a strictly decreasing, positive, null sequence, the sequence of sets $(\delta_n\mathbb{Z})$ is a sequence of embeddings of $\mathbb{Z}$ in $\mathbb{R}$. Similarly, if (τ_n) is a strictly decreasing, positive, null sequence, the sequence of sets $(\tau_n\mathbb{N}_0)$ is a sequence of embeddings of $\mathbb{N}_0$ in the non-negative real numbers. The aforementioned maps make it possible to view the primitive random walk as modeling the position of a particle moving in time throughout a grid in $\mathbb{R}$. A real spatiotemporal embedding with parameter (τ_n, δ_n) is a map ι_n with

$$\iota_n\colon \mathbb{N}_0 \times \mathbb{Z} \to \mathbb{R}\times\mathbb{R} \quad \text{by} \quad \iota_n(m, z) = (\tau_n m, \delta_n z).$$

The embedding ι_n induces a mapping from the random process S to a stochastic process indexed by $\mathbb{R}_{\geq 0}$ and valued in $\mathbb{R}$, a spatiotemporal embedding of S denoted by $\iota_n S$ and initially defined only abstractly.

For each non-negative t and natural number n, define the random variable

$$\tilde{Y}^{(n)}_t = \delta_n\left(\tilde{X}_0 + \cdots + \tilde{X}_{\left\lfloor \frac{t}{\tau_n}\right\rfloor}\right).$$

Define by $F([0,\infty)\colon \mathbb{R})$ the space of all real valued functions from $[0,\infty)$ to $\mathbb{R}$. For any history h and an associated cylinder set $C(h)$ of $F([0,\infty)\colon \mathbb{R})$, define $P^{(n)}(C(h))$ in the following way. If for each i in $\{1,\dots,\ell(h)\}$,

$$t_i = e(h)_i, \quad U_i = U(h)_i, \quad \text{and} \quad \ell(h) = \ell$$

then

$$P^{(n)}(C(h)) = \sum_{x_0\in U_0\cap\delta_n\mathbb{Z}}\sum_{x_1\in U_1\cap\delta_n\mathbb{Z}}\cdots\sum_{x_\ell\in U_\ell\cap\delta_n\mathbb{Z}} \mathrm{Prob}\left(\tilde{Y}^{(n)}_0 = x_0\right)\mathrm{Prob}\left(\tilde{Y}^{(n)}_{t_1} = x_1 - x_0\right)$$
$$\cdot\, \mathrm{Prob}\left(\tilde{Y}^{(n)}_{t_2} - \tilde{Y}^{(n)}_{t_1} = x_2 - x_1\right)\cdots\mathrm{Prob}\left(\tilde{Y}^{(n)}_{t_\ell} - \tilde{Y}^{(n)}_{t_{\ell-1}} = x_\ell - x_{\ell-1}\right).$$

The Kolmogorov Extension theorem implies that the pre-measure defined in this way on cylinder sets extends to a measure, once again denoted by $P^{(n)}$, on all of $F([0,\infty)\colon \mathbb{R})$.

Define for each non-negative t the random variable $Y_t^{(n)}$ acting on the sample space $\big(F([0,\infty)\colon \mathbb{R}), P^{(n)}\big)$ by

$$Y_t^{(n)}(\omega) = \omega(t).$$

The stochastic process $Y^{(n)}$ with

$$Y^{(n)}\colon t \mapsto Y_t^{(n)}$$

is a concrete model for the process $\tilde{Y}^{(n)}$. The set $F([0,\infty)\colon \mathbb{R})$ is too large of a set and with elements that are too poorly behaved to be useful for addressing analytical questions and it is important to find a version of the process in a more restrictive sample space.

Suppose that $\tilde{Z}$ is an abstractly defined stochastic process such that for each t in a time interval I, $\tilde{Z}_t$ is an abstractly defined random variable taking values in a Polish space $\mathcal{S}$. Suppose that M is a probability measure on $F(I\colon \mathcal{S})$ and that the probability space $(F(I\colon \mathcal{S}), M)$ together with the stochastic process Z is a concrete model for $\tilde{Z}$. If there are positive constants a, b, C and ε such that for any epoch (t_1, t_2, t_3) of I,

$$\mathrm{E}\Big[|Z_{t_2} - Z_{t_1}|^a\,|Z_{t_3} - Z_{t_2}|^b\Big] \le C\,(t_3 - t_1)^{1+\varepsilon},$$

then the stochastic process Z has a version in the space $D(I\colon \mathcal{S})$, [9, 10]. If Z satisfies this moment estimate, Centsov's criterion, then the measure associated to Z is tight. Given a sequence $\big(Z^{(n)}\big)$ of stochastic processes with sample paths in the path space $D(I\colon \mathbb{Q}_p)$, if this estimate holds for each of the $Z^{(n)}$ in the sequence with the constants the same for each n and for any epoch of three time points in I, then the set of probability measures $\{M_n\colon n \in \mathbb{N}\}$ given by the sequence $\big(Z^{(n)}\big)$ is uniformly tight in the space of measures on $D(I\colon \mathbb{Q}_p)$. This is to say that for any positive ε, there is a compact subset K of $D(I\colon \mathbb{Q}_p)$ such that for each n,

$$M_n(K) > 1 - \varepsilon.$$

If the finite dimensional distributions of the M_n converge those of M, then uniform tightness of the collection of measures $\{M_n\colon n \in \mathbb{N}\}$ implies the weak-$*$ convergence of the sequence of measures (M_n) to the measure M. In the setting of discrete time random walks converging to a real Brownian motion, if I is compact and (δ_n) and (τ_n) are chosen so that there is a constant K with

$$\frac{\delta_n^2}{\tau_n} \to K,$$

then there is such a uniform moment estimate for the $P^{(n)}$. Together with the convergence of the $P^{(n)}$ to the Wiener measure W, this uniform estimate proves the weak-$*$ convergence of the $P^{(n)}$ to W in the Skorohod space $D(I\colon \mathbb{R})$.

The current paper follows this framework closely but difficulties arise in key places. The choice of the primitive random walk in the real setting is entirely classical, but the choice of a primitive random walk in the p-adic setting that Section 4 presents seems to be the first new idea of the paper. Just as in the real case, spatiotemporal embeddings of the primitive process give rise to concrete processes taking values in the Skorohod space on

$\mathbb{Q}_p$. However, determining the spatiotemporal embeddings and the relationship between the time and distance scales, which bear striking similarity to those in the real setting, is slightly more difficult in the p-adic setting. For each of the approximating process, Section 5 introduces two distinct versions, one with sample paths in $\mathbb{Q}_p$ that are the focus of the paper and one in a discrete quotient group where calculations are easier to justify. In the real setting, the convergence of the finite dimensional distributions of the discrete time processes presents some difficulty but is a consequence of the de Moivre–Laplace theorem. Uniform tightness of the measures in the real case is, however, easy to prove. In contrast, the proof of convergence of the finite dimensional distributions is arguably much easier in the p-adic setting. However, the proof of the tightness of the measures in the p-adic setting is much more difficult than in the real setting. The principle *technical* challenge of the current paper is to obtain uniformity in the moment estimate, which Section 6 achieves.

4. The Primitive Process

The result of this section and the next is the construction of a family of stochastic processes, each with sample paths in the Skorohod space of paths valued in $\mathbb{Q}_p$ that almost surely take values in a discrete subset of $\mathbb{Q}_p$ and that are almost surely step functions that, for some positive τ, are constant on every interval in $[0,\infty)\setminus\tau\mathbb{N}$. To this end, we first construct an abstract primitive process and find a concrete version of this process that we refer to as the primitive process. Denote by $\mathbb{N}_0$ the natural numbers with zero included. In the next section, we define families of spatiotemporal embeddings where the embeddings map the primitive process to a process with paths in $D([0,\infty)\colon \mathbb{Q}_p)$ or, for some n in $\mathbb{N}_0$, to $D([0,\infty)\colon \mathbb{Q}_p/p^n\mathbb{Z}_p)$.

Denote by G the group $\mathbb{Q}_p/\mathbb{Z}_p$. Let $[\cdot]$ be the quotient map from $\mathbb{Q}_p$ onto $\mathbb{Q}_p/\mathbb{Z}_p$ and define a modulus $|\cdot|$ on G by

$$|[x]| = \begin{cases} |x| & \text{if } [x]\neq[0] \\ 0 & \text{if } [x]=[0]. \end{cases}$$

For each n in $\mathbb{N}_0$, define by C_n the set

$$C_n = \{[x]\in G\colon |[x]| = p^n\}.$$

Visualize G as an atom (Figure 1), with each p^n a principle energy level and each element of C_n a particular state having principle quantum number given by the shell C_n. Of course, the analogy here is given purely for visual aesthetic and intuition. The primitive random walk will describe a particle that jumps between the "electron states" of this "atom." Fix b to be larger than one and let $\tilde{X}$ be a random variable with distribution

$$\operatorname{Prob}\big(\tilde{X} = [\mathbb{Z}_p]\big) = \frac{p^b-2}{p^b-1} \quad \text{and} \quad \operatorname{Prob}\big(\tilde{X}\in C_n\big) = \frac{1}{p^{bn}}$$

and such that, for each n, the probability mass function for $\tilde{X}$ is uniform when restricted to C_n. Denote by $\mathrm{vol}(C_n)$ the measure of C_n. The condition of uniformity implies that for each $[x]$ in C_n,

$$\mathrm{Prob}\big(\tilde{X} = [x]\big) = \frac{1}{\mathrm{vol}(C_n)} \cdot \frac{1}{p^{bn}} = \frac{1}{p^n - p^{n-1}} \cdot \frac{1}{p^{bn}}.$$

Sum the geometric series below to obtain the equality

$$\mathrm{Prob}\big(\tilde{X} \in G\big) = \frac{p^b - 2}{p^b - 1} + \frac{1}{p^b} + \frac{1}{p^{2b}} + \frac{1}{p^{3b}} + \cdots = 1.$$

Therefore, the non-negative function g given by

$$g([x]) = \mathrm{Prob}(\tilde{X} = [x])$$

is a probability mass function on G.

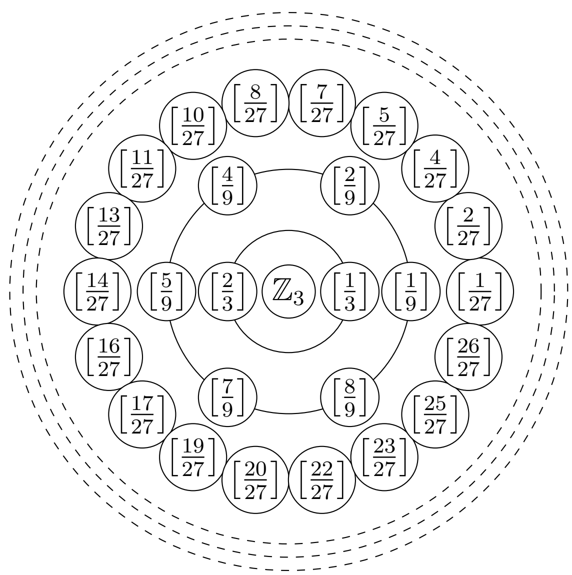

Figure 1. The State Space $\mathbb{Q}_3/\mathbb{Z}_3$ as an Atom

Let $\big(\tilde{X}_i\big)_{i\in\mathbb{N}}$ be a sequence of independent identically distributed random variables, each with the same distribution as $\tilde{X}$. Let $\tilde{X}_0$ be a random variable equal to $[0]$ with probability one. Define by $\tilde{S}_n$ the random variable

$$\tilde{S}_n = \tilde{X}_0 + \tilde{X}_1 + \cdots + \tilde{X}_n$$

and by $\tilde{S}$ the stochastic process

$$\tilde{S} \colon n \mapsto \tilde{S}_n.$$

Define by $F(\mathbb{N}_0\colon G)$ the set of all functions that map $\mathbb{N}_0$ into G. Let h be a history for S given by

$$h = \left((0, U_0), (t_1, U_1), \ldots, (t_k, U_k)\right).$$

Define a pre-measure, P, on the cylinder sets of S by

$$P(C(h)) = \sum_{x_0 \in U_0} \cdots \sum_{x_k \in U_k} \mathrm{Prob}(\tilde{S}_0 = x_0)\mathrm{Prob}(\tilde{S}_1 - \tilde{S}_0 = x_1 - x_0) \\ \cdots \mathrm{Prob}(\tilde{S}_k - \tilde{S}_{k-1} = x_k - x_{k-1}).$$

The Kolmogorov Extension theorem guarantees the existence of a measure, to be again denoted by P, on the set $F(\mathbb{N}_0\colon G)$ that restricts to the above pre-measure on the cylinder sets. Define for each n the random variable S_n by

$$S_n(\omega) = \omega(n) \quad \text{where} \quad \omega \in F(\mathbb{N}_0\colon G).$$

The sequence, S, of random variables (S_n) acting on the probability space $(F(\mathbb{N}_0\colon G), P)$ is a discrete time stochastic process and a concrete model for the process $\tilde{S}$. Note that this sample space together with the stochastic process S gives a model for the $\tilde{X}_i$. In particular, for each i in $\mathbb{N}_0$, the random variable X_i associated to $\tilde{X}_i$ acts on paths by

$$X_0(\omega) = S_0(\omega) \quad \text{and} \quad X_{i+1}(\omega) = (S_{i+1} - S_i)(\omega).$$

The X_i are the increments of the process S and are independent and identically distributed, so that S is a sum of independent identically distributed random variables.

Proposition 1. *For each natural number n, the real-valued random variable $|S_n|$ has bounded moments. In particular, for each positive real number k strictly less than b, the moments of $|S_n|$ satisfy the inequality*

$$\mathrm{E}\left[|S_n|^k\right] \le \frac{n}{p^{b-k} - 1}.$$

Proof. Let k be a positive real number. The random variable $|X|$ has k^{th} moment

$$\begin{aligned}\mathrm{E}\left[|X|^k\right] &= \mathrm{E}\left[|\tilde{X}|^k\right] \\ &= 0^k \frac{p-2}{p-1} + p^k \frac{1}{p^b} + p^{2k} \frac{1}{p^{2b}} + \cdots = \frac{1}{p^{b-k} - 1}.\end{aligned}$$

The ultrametric property of $|\cdot|$ implies that

$$|X_1 + \cdots + X_n| \le \max_{1 \le i \le n} |X_i|$$

and so

$$\begin{aligned}\mathrm{E}\Big[|S_n|^k\Big] &= \mathrm{E}\Big[|X_1+\cdots+X_n|^k\Big]\\ &\le \mathrm{E}\left[\left(\max_{1\le i\le n}|X_i|\right)^k\right]\\ &\le \mathrm{E}\Big[|X_1|^k+\cdots+|X_n|^k\Big] = \frac{n}{p^{b-k}-1}.\end{aligned}$$

□

5. Spatiotemporal Embeddings

For each x in $\mathbb{Q}_p$, recall that there is a unique function a_x on $\mathbb{Z}$ such that a_x is zero for all but finitely many negative integers and

$$x = \sum_{k\in\mathbb{Z}} a_x(k)p^k.$$

For each n in $\mathbb{N}_0$, let

$$G_n = \mathbb{Q}_p/p^n\mathbb{Z}_p.$$

Of course, G_0 is equal to G. For each n in $\mathbb{N}_0$, denote by $[\cdot]_n$ the quotient map from $\mathbb{Q}_p$ onto G_n. Define a modulus $|\cdot|$ on G_n by

$$|[x]_n| = \begin{cases}|x| & \text{if } [x]_n \neq [0]_n\\ 0 & \text{if } [x]_n = [0]_n.\end{cases}$$

For each n in $\mathbb{Z}$ and x in G_n, define by $B_n(x)$ the set

$$B_n(x) = \{y\in G_n\colon |y-x|\le p^n\}.$$

Define an injection j_n of G_n into $\mathbb{Q}_p$ in the following way. For each y in G_n, there is an x in $\mathbb{Q}_p$ such that

$$y = [x]_n.$$

Pick for each y a particular such x, denoted by $]y[_n$. The map

$$]\cdot[_n\colon G_n\to\mathbb{Q}_p$$

is an injection of G_n into $\mathbb{Q}_p$. Naturally, there are many such injections and there is no a priori reason to prefer one such injection over another. For the sake of simplifying the exposition, it is preferable to work in a more consistent albeit unnecessarily restrictive framework. Denote by $\mathbb{1}_{(-\infty,n)}$ the characteristic function on $(-\infty,n)\cap\mathbb{Z}$ and, requiring that y vary in the set G_n, define j_n by

$$j_n(y) = \sum_{k\in\mathbb{Z}} a_{]y[_n}(k)\,\mathbb{1}_{(-\infty,n)}(k)p^k.$$

Of course, the injection j_n is not dependent on the choices imposed by the injection $]\cdot[_n$. Note that each $j_n(G_n)$ inherits an additive group structure from the additive group structure on G_n, although this group structure does not agree with the group structure on $\mathbb{Q}_p$. Denote by α_n the group isomorphism

$$\alpha_n\colon G \to G_n \quad \text{by} \quad \alpha_n(x + \mathbb{Z}_p) = p^n x + p^n\mathbb{Z}_p \quad \text{for} \quad x \in \mathbb{Q}_p.$$

An advantage of the consistency of the given choice of injection is that for any y in G,

$$j_n(\alpha_n(y)) = p^n j_0(y).$$

Definition 7. Let l be a strictly increasing function with

$$l\colon \mathbb{N}_0 \to [1, \infty),$$

such that $l(0)$ equals one. A *sequence of p-adic spatiotemporal embeddings with time scaling* l is a sequence ι, where for each n,

$$\iota_n\colon \mathbb{N}_0 \times G \to [0, \infty) \times \mathbb{Q}_p \quad \text{by} \quad \iota_n(m, [x]) = \left(\frac{m}{l(n)}, p^n j_0([x])\right).$$

A *sequence of* G_n *spatiotemporal embeddings with time scaling* l is a sequence $\iota^\sharp$, where

$$\iota_n^\sharp\colon \mathbb{N}_0 \times G \to [0, \infty) \times G_n \quad \text{by} \quad \iota_n(m, [x]) = \left(\frac{m}{l(n)}, \alpha_n([x])\right).$$

The embeddings ι_n and $\iota_n^\sharp$ induce maps on the random process S, respectively mapping S to a stochastic process $\iota_n S$ that is indexed by $[0, \infty)$ and valued in $\mathbb{Q}_p$ and to a stochastic process $\iota_n^\sharp S$ that is indexed by $[0, \infty)$ and valued in G_n. To construct $\iota_n S$, we first construct an abstract stochastic processes that can be viewed as spatiotemporal embedding of $\tilde{S}$ and then choose a specific concrete model for the abstractly defined process, which we view as the spatiotemporal embedding of S. We follow the same procedure to construct $\iota_n^\sharp S$. We will further show that the range of a sample path for the process $\iota_n S$ is almost surely a subset of $j_n(G_n)$.

The distance scale δ and time scale τ of a sequence of space-temporal embeddings ι and $\iota^\sharp$ are themselves sequences and their n^{th} places respectively equal the factor by which distance increments and time increments change for the given sequence of embeddings at the same place. The distance scale, $\delta(n)$, is the same for both ι_n and $\iota_n^\sharp$ is equal to p^{-n}. The time scale, $\tau(n)$, is the same for both embeddings and is equal to $\frac{1}{l(n)}$. The relationship between $\delta(n)$ and $\tau(n)$ will be an important determination to be made in Section 5.4.

5.1. **Embeddings with $\mathbb{Q}_p$-Valued Paths.** For each non-negative real t, define abstractly the random variable

$$\tilde{Y}_t^{(n)} = p^n j_0\big(\tilde{S}_{\lfloor t\, l(n)\rfloor}\big).$$

We will now find a model for $\tilde{Y}^{(n)}$ in an appropriate space of $\mathbb{Q}_p$-valued paths. Let h be a history with an epoch $e(h)$ of length ℓ and a route $U(h)$. Define the probability of the cylinder set $C(h)$ to be $P_n(C(h))$ where

$$P_n(C(h)) = \sum_{x_0 \in U(h)_0 \cap j_n(G_n)} \cdots \sum_{x_\ell \in U(h)_\ell \cap j_n(G_n)} \mathrm{Prob}\big(\tilde{Y}_0 = x_0\big) \cdot \mathrm{Prob}\big(\tilde{Y}_{t_1} - \tilde{Y}_0 = x_1 - x_0\big) \cdot \mathrm{Prob}\big(\tilde{Y}_{t_2} - \tilde{Y}_{t_1} = x_2 - x_1\big) \cdot \cdots \cdot \mathrm{Prob}\big(\tilde{Y}_{t_\ell} - \tilde{Y}_{t_{\ell-1}} = x_\ell - x_{\ell-1}\big).$$

The Kolmogorov Extension theorem implies that these pre-measures extend to a measure, that we again denote by P_n, on $F([0,\infty)\colon \mathbb{Q}_p)$. Define for each non-negative t the random variable $Y_t^{(n)}$ acting on the sample space $F([0,\infty)\colon \mathbb{Q}_p)$ by

$$Y_t^{(n)}(\omega) = \omega(t).$$

The stochastic process $Y^{(n)}$ with

$$Y^{(n)}\colon t \mapsto Y_t^{(n)}$$

is a concrete model for the process $\tilde{Y}^{(n)}$ and is a sum of independent increments. The sample space $F([0,\infty)\colon \mathbb{Q}_p)$ is too large of a space to be very useful and it behooves us to find a version of the process with a smaller sample space. To this end, we will prove the following moment estimate.

Lemma 1. *For each positive real number k strictly less than b and each non-negative t, the real valued random variable $\big|Y_t^{(n)}\big|$ satisfies the moment estimate*

$$\mathrm{E}\Big[\big|Y_t^{(n)}\big|^k\Big] \leq \frac{p^{-kn}}{p^{b-k}-1} l(n)t.$$

Proof. Proposition 1 implies that

$$\begin{aligned} \mathrm{E}\Big[\big|Y_t^{(n)}\big|^k\Big] &= \mathrm{E}\Big[\Big|p^n j_0\left(\tilde{S}_{\lfloor t\, l(n)\rfloor}\right)\Big|^k\Big] \\ &= p^{-kn}\mathrm{E}\Big[\Big|j_0\left(\tilde{S}_{\lfloor tl(n)\rfloor}\right)\Big|^k\Big] \\ &= p^{-kn}\mathrm{E}\Big[\big|\tilde{S}_{\lfloor tl(n)\rfloor}\big|^k\Big] \\ &\leq \frac{p^{-kn}}{p^{b-k}-1}\lfloor tl(n)\rfloor \leq \frac{p^{-kn}}{p^{b-k}-1} l(n)t. \end{aligned}$$

□

Proposition 2. *There is a version of this stochastic process $Y^{(n)}$ that has sample paths in the Skorohod Space $D([0,\infty)\colon \mathbb{Q}_p)$.*

Proof. Suppose that (t_1, t_2, t_3) is an epoch and k is in $(0, b)$. Since $Y^{(n)}$ has independent increments, Lemma 1 implies that

$$\begin{aligned}\mathrm{E}\left[\left|Y^{(n)}_{t_2} - Y^{(n)}_{t_1}\right|^k \left|Y^{(n)}_{t_3} - Y^{(n)}_{t_2}\right|^k\right] &= \mathrm{E}\left[\left|Y^{(n)}_{t_2} - Y^{(n)}_{t_1}\right|^k\right] \mathrm{E}\left[\left|Y^{(n)}_{t_3} - Y^{(n)}_{t_2}\right|^k\right] \\ &\leq \frac{p^{-kn}}{p^{b-k}-1} l(n)(t_2 - t_1) \frac{p^{-kn}}{p^{b-k}-1} l(n)(t_3 - t_2) \\ &< \left(\frac{p^{-kn}}{p^{b-k}-1} l(n)\right)^2 (t_3 - t_1)^2.\end{aligned}$$

Since $Y^{(n)}$ satisfies Centsov's criterion, there is a version of this stochastic process that has paths in $D([0,\infty)\colon \mathbb{Q}_p)$ [10]. Denote this version again by $Y^{(n)}$. Henceforth, the notation $Y^{(n)}$ will exclusively refer to this version. □

The process $\iota_n S$ is defined to be this process $Y^{(n)}$, a spatiotemporal embedding of the process S.

Proposition 3. *The subset of paths in $D([0,\infty)\colon \mathbb{Q}_p)$ for the process $Y^{(n)}$ that, for each natural number m, are constant on the intervals I_m with*

$$I_m = \left[\frac{m-1}{l(n)}, \frac{m}{l(n)}\right)$$

and that are $p^n j_0(G)$ valued have full measure.

Proof. For any epoch $(t_1, \ldots, t_k)$, the set of paths in $D([0,\infty)\colon \mathbb{Q}_p)$ specified to be in $p^n j_0(G)$ at any given place in the epoch is P_n-almost sure. Finite intersections of almost sure events are almost sure and so

$$P_n(\{\omega \in D([0,\infty)\colon \mathbb{Q}_p)\colon \omega(t_i) \in p^n j_0(G), 1 \leq i \leq k\}) = 1.$$

There is a collection $\{A_i\colon i \in \mathbb{N}\}$ of strictly increasing nested finite subsets of $\mathbb{Q}$ whose union is $\mathbb{Q}$. Therefore,

$$\begin{aligned}&P_n\left(\{\omega \in D([0,\infty)\colon \mathbb{Q}_p)\colon \omega(s) \in p^n j_0(G), \forall s \in \mathbb{Q}\}\right) \\ &\quad = \lim_{i\to\infty} P_n(\{\omega \in D([0,\infty)\colon \mathbb{R})\colon \omega(s) \in p^n j_0(G), \forall s \in A_i\}) = 1.\end{aligned}$$

The right continuity of the paths implies that the set of $j_n(G_n)$-valued paths is P_n-almost sure.

To prove that the paths can only change values at time points in the set T_n where

$$T_n = \left\{\frac{m}{l(n)}\colon m \in \mathbb{N}\right\},$$

suppose that t is a point that is not in T_n. In this case, there is a natural number m such that t is an interior point of I_m. Let (V_i) be an increasing sequence of finite subsets of I_m

with the property that

$$\bigcup_{i\in\mathbb{N}} V_i = I_m \cap \mathbb{Q}.$$

For any pair of real numbers s_1 and s_2, if $\lfloor s_1 l(n)\rfloor$ equals $\lfloor s_2 l(n)\rfloor$, then

$$\begin{aligned}\operatorname{Prob}\big(Y^{(n)}_{s_1} - Y^{(n)}_{s_2} = 0\big) &= \operatorname{Prob}\big(\tilde{Y}^{(n)}_{s_1} - \tilde{Y}^{(n)}_{s_2} = 0\big)\\ &= \operatorname{Prob}\big(\tilde{S}_{\lfloor s_1 l(n)\rfloor} - \tilde{S}_{\lfloor s_2 l(n)\rfloor} = 0\big) = 1.\end{aligned}$$

Therefore,

$$\begin{aligned}&P_n(\{\omega \in D([0,\infty)\colon \mathbb{Q}_p)\colon |\omega(t) - \omega(s)| = 0, \quad \forall s \in I_m \cap \mathbb{Q}\})\\ &= \lim_{i\to\infty} P_n(\{\omega \in D([0,\infty)\colon \mathbb{Q}_p)\colon |\omega(t) - \omega(s)| = 0, \quad \forall s \in I_m \cap V_i\}) = 1.\end{aligned}$$

The right continuity of the paths implies that

$$P_n(\{\omega \in D([0,\infty)\colon \mathbb{Q}_p)\colon |\omega(t) - \omega(s)| = 0, \quad \forall s \in I_m\}) = 1,$$

thus proving the proposition. □

5.2. **The G_n-Valued Version.** While we ultimately work in the setting of $\mathbb{Q}_p$-valued paths, certain calculations are easier to perform and justify in a discrete setting. To this end, we construct a sequence of G_n-valued spatiotemporal embeddings of the process S. Once again, begin with an abstractly defined random variable denoted by $\tilde{Y}^{(n),\sharp}_t$, where

$$\tilde{Y}^{(n),\sharp}_t = \alpha_n\big(\tilde{S}_{\lfloor t\, l(n)\rfloor}\big).$$

If h is a history for paths valued in G_n and k is the length of the epoch $e(h)$, then define the probability of the cylinder set $C(h)$ to be $P^\sharp_n(C(h))$ where

$$\begin{aligned}P^\sharp_n(C(h)) = \sum_{x_0\in U(h)_0} \cdots \sum_{x_k\in U(h)_k} &\operatorname{Prob}\big(\tilde{Y}^{(n),\sharp}_0 = x_0\big)\\ &\cdot \operatorname{Prob}\big(\tilde{Y}^{(n),\sharp}_{t_1} - \tilde{Y}^{(n),\sharp}_0 = x_1 - x_0\big) \cdot \operatorname{Prob}\big(\tilde{Y}^{(n),\sharp}_{t_2} - \tilde{Y}^{(n),\sharp}_{t_1} = x_2 - x_1\big)\\ &\cdot\ \cdots\ \cdot \operatorname{Prob}\big(\tilde{Y}^{(n),\sharp}_{t_k} - \tilde{Y}^{(n),\sharp}_{t_{k-1}} = x_k - x_{k-1}\big).\end{aligned}$$

The Kolmogorov Extension theorem implies that these pre-measures extend to a measure on all of $F([0,\infty)\colon G_n)$, once again to be denoted by $P^\sharp_n$. If x is in G, then

$$|\alpha_n(x)| = p^{-n}|x|.$$

The equality above and the ultrametricity of the metric together imply the moment estimates that ensure that there is a version of this process in $D([0,\infty)\colon G_n)$. The similarity of the calculations to those performed in Lemma 1 and Proposition 2 prompts their present omission. Once again denote by $P^\sharp_n$ the measure on $D([0,\infty)\colon G_n)$ that takes the above values on cylinder sets. Define for each non-negative t the random variable $Y^{(n),\sharp}_t$ by

$$Y^{(n),\sharp}_t \colon D([0,\infty)\colon G_n) \to G_n \quad \text{by} \quad Y^{(n),\sharp}_t(\omega) = \omega(t)$$

and define the process $Y^{(n),\sharp}$ by

$$Y^{(n),\sharp}\colon t \to Y_t^{(n),\sharp}.$$

The G_n-valued spatiotemporal embedding of the process S, $\iota_n^\sharp S$, is defined to be the process $Y_t^{(n),\sharp}$.

Proposition 4. *The subset of paths in $D([0,\infty)\colon G_n)$ for the process $Y^{(n),\sharp}$ that, for each natural number m, are constant on the intervals I_m with*

$$I_m = \left[\frac{m-1}{l(n)}, \frac{m}{l(n)}\right)$$

have full measure.

The similarity of the proof of Proposition 4 to that of Proposition 3 prompts its omission here. If ω is a path in $D([0,\infty)\colon \mathbb{Q}_p)$, then $[\omega]_n$ is a path in $D([0,\infty)\colon G_n)$. The proof that $[\cdot]_n$ is an isometric map between $j_n(G_n)$ and G_n is straightforward. If A is a subset of $D([0,\infty)\colon \mathbb{Q}_p)$, then define $[A]_n$ to be the set of all $[\omega]_n$ where ω is in A. Suppose that A is a subset of $D([0,\infty)\colon \mathbb{Q}_p)$ and that A_n is equal to $A \cap D([0,\infty)\colon j_n(G_n))$. If A is a cylinder set, then so is A_n and so

$$P_n(A) = P_n(A_n) = P_n^\sharp([A_n]_n).$$

This equality will hold for any measurable subset of $D([0,\infty)\colon \mathbb{Q}_p)$ by the uniqueness of the extension of the measures to the σ-algebras they generate and since $[A_n]_n$ is as well a measurable set. The spatiotemporal imbedding $\iota_n^\sharp$ gives a way to make computations in the G_n setting rather than in the $\mathbb{Q}_p$ setting. Some calculations will be more easily justified in this discrete setting.

5.3. **Duality and the Discrete Groups.** For each n, the group $p^{-n}\mathbb{Z}_p$ is the Pontryagin dual of the group G_n. If $[x]_n$ is in G_n and y is in $p^{-n}\mathbb{Z}_p$, viewed via the inclusion map as a subset of $\mathbb{Q}_p$, then define the dual pairing $\langle\cdot,\cdot\rangle_n$ with

$$\langle\cdot,\cdot\rangle_n\colon p^{-n}\mathbb{Z}_p \times G_n \to S^1 \quad \text{by} \quad \langle [x]_n, y\rangle_n = \chi(xy).$$

While the definition of the dual pairing uses a specific representative of the equivalence class, it is independent of the choice of the representative. In particular, if $[z]_n$ is equal to $[x]_n$, then there is an α in $p^n\mathbb{Z}_p$ such that

$$z = x + \alpha,$$

hence

$$\begin{aligned}\chi(zy) &= \chi((x+\alpha)y)\\ &= \chi(xy)\chi(\alpha y) = \chi(xy),\end{aligned}$$

because αy is in $\mathbb{Z}_p$ and χ is a rank zero character on $\mathbb{Q}_p$. To compress notation, henceforth suppress the use of n in the notation for the dual pairing.

Define the Fourier transform $\mathcal{F}_n$ as the unitary extension to all of $L^2(G_n)$ of the map that is unitary under the L^2 norm and initially defined from $L^1(G_n) \cap L^2(G_n)$ to $L^2(p^{-n}\mathbb{Z}_p)$ by

$$(\mathcal{F}_n f)(y) = \int_{G_n} \langle -[x]_n, y\rangle \, \mathrm{d}[x]_n.$$

The inverse Fourier transform, denoted by $\mathcal{F}_n^{-1}$, is defined on all of $L^2(p^{-n}\mathbb{Z}_p)$ but is given by

$$\left(\mathcal{F}_n^{-1} f\right)([x]_n) = \int_{p^{-n}\mathbb{Z}_p} \langle [x]_n, y\rangle \, \mathrm{d}y$$

when restricted to $L^1(p^{-n}\mathbb{Z}_p) \cap L^2(p^{-n}\mathbb{Z}_p)$.

5.4. **Determination of the Time Scale.** For each natural number i, define by $X_i^{(n),\sharp}$ the random variable

$$X_i^{(n),\sharp} = Y^{(n),\sharp}_{\left\lfloor \frac{i}{l(n)} \right\rfloor} - Y^{(n),\sharp}_{\left\lfloor \frac{i-1}{l(n)} \right\rfloor} \quad \text{and} \quad X_0^{(n),\sharp} = Y_0^{(n),\sharp}.$$

Proposition 4 guarantees that the process $Y^{(n),\sharp}$ is almost surely a sum of the independent identically distributed increments given by the $X_i^{(n),\sharp}$, in particular,

$$Y_t^{(n),\sharp} \overset{\text{a.s.}}{=} X_0^{(n),\sharp} + X_1^{(n),\sharp} + \cdots + X^{(n),\sharp}_{\lfloor tl(n) \rfloor}.$$

Denote by $g^{(n),\sharp}$ the common probability density function (in this case a mass function) for all of the random variables $X_i^{(n),\sharp}$. The function $g^{(n),\sharp}$ is a function acting on G_n. For each positive t, denote by $f_t^{(n),\sharp}$ the probability density function for $Y_t^{(n),\sharp}$. If U is a Borel subset of $\mathbb{Q}_p$ then

$$\begin{aligned}\operatorname{Prob}\big(Y_t^{(n)} \in U\big) &= \operatorname{Prob}\big(Y_t^{(n)} \in U \cap j_n(G_n)\big) \\ &= \operatorname{Prob}\big(Y_t^{(n),\sharp} \in [U \cap j_n(G_n)]_n\big) = \int_{[U\cap j_n(G_n)]_n} f_t^{(n),\sharp}([x]_n) \, \mathrm{d}[x]_n.\end{aligned}$$

Since the probability density function of a sum of random variables is the convolution of their density function,

$$f_t^{(n),\sharp}([x]_n) = \underbrace{\big(g^{(n),\sharp} * \cdots * g^{(n),\sharp}\big)}_{\lfloor tl(n) \rfloor \text{ times}}([x]_n).$$

The Fourier transform takes convolution to multiplication and so

$$\big(\mathcal{F}_n f_t^{(n),\sharp}\big)([x]_n) = \big(\mathcal{F}_n g^{(n),\sharp}([x]_n)\big)^{\lfloor tl(n) \rfloor}.$$

Note that if $|[x_1]_n|$ and $|[x_2]_n|$ are equal, then $g^{(n),\sharp}([x_1]_n)$ and $g^{(n),\sharp}([x_2]_n)$ are equal. Recall that

$$\int_{|[x]_n| = p^k} \langle [x]_n, a\rangle \, \mathrm{d}[x]_n = \begin{cases} p^k\left(1 - \frac{1}{p}\right) & \text{if } |a| \le p^{-k} \\ -p^{k-1} & \text{if } |a| = p^{-k+1} \\ 0 & \text{if } |a| \ge p^{-k+2}. \end{cases}$$

The calculations parallel those in the $\mathbb{Q}_p$ setting [23]. Note that for the equation above a is in $p^{-n}\mathbb{Z}_p$ and so $|a|$ is at most p^n and furthermore that k is greater than $-n$.

For any i in $\mathbb{N}_0$, define the quantities g_i by

$$g_0 = g^{(n),\sharp}([0]_n) = \frac{p^b-2}{p^b-1}\frac{1}{\operatorname{vol}\left(\alpha_n\left(\mathbb{Z}_p\right)\right)} \quad \text{and} \quad g_i = g^{(n),\sharp}([p^{n-i}]_n) = \frac{1}{p^{ib}}\frac{1}{\operatorname{vol}\left(\alpha_n\left(C_i\right)\right)}.$$

To calculate $\left(\mathcal{F}_n f_t^{(n),\sharp}\right)(y)$, where necessarily y is in $p^{-n}\mathbb{Z}_p$, note that

$$\begin{aligned}\left(\mathcal{F}_n g^{(n),\sharp}\right)(y) &= \int_{G_n} \langle [x]_n, y\rangle g^{(n),\sharp}([x]_n)\,\mathrm{d}[x]_n \\ &= g_0 \int_{\alpha_n(Z_p)} \langle [x]_n, y\rangle\,\mathrm{d}[x]_n + g_1 \int_{\alpha_n(C_1)} \langle [x]_n, y\rangle\,\mathrm{d}[x]_n \\ &\quad + \cdots + g_m \int_{\alpha_n(C_m)} \langle [x]_n, y\rangle\,\mathrm{d}[x]_n + \cdots .\end{aligned}$$

If k is once again be larger than $-n$ and if y is such that

$$|y| = p^k,$$

then

$$\int_{C_m} \langle [x]_0, y\rangle\,\mathrm{d}[x]_0 = \begin{cases} 0 & \text{if } p^k \geq p^{2-m} \\ -p^{m-1} & \text{if } p^k = p^{1-m} \\ \operatorname{vol}\left(C_m\right) & \text{otherwise.}\end{cases}$$

This equality generalizes to n in $\mathbb{N}_0$ as

$$\int_{\alpha_n(C_m)} \langle [x]_n, y\rangle\,\mathrm{d}[x]_n = \begin{cases} 0 & \text{if } p^k \geq p^{2+n-m} \\ -p^{m-n-1} & \text{if } p^k = p^{1+n-m} \\ \operatorname{vol}\left(\alpha_n\left(C_m\right)\right) & \text{otherwise,}\end{cases}$$

and so

$$\begin{aligned}&\int_{\alpha_n(C_{-k+n+1})} g^{(n),\sharp}\left([x]_n\right)\langle [x]_n, y\rangle\,\mathrm{d}[x]_n \\ &\qquad = \frac{1}{\operatorname{vol}\left(\alpha_n\left(C_{-k+n+1}\right)\right)}\frac{1}{p^{(-k+n+1)b}} \int_{\alpha_n(C_{-k+n+1})} \langle [x]_n, y\rangle\,\mathrm{d}[x]_n \\ &\qquad = \frac{1}{p^{(-k+n+1)b}}\frac{1}{p^{-k+1}\left(1-\frac{1}{p}\right)}\left(-p^{-k}\right) \\ &\qquad = -\frac{p^{kb}}{p^{b+1}\left(1-\frac{1}{p}\right)p^{nb}} = -\frac{1}{p^b(p-1)}\frac{|y|^b}{p^{nb}}. \end{aligned} \tag{4}$$

Use (4) to obtain the equalities

$$
\begin{aligned}
(\mathcal{F}_n g^{(n),\sharp})(y) &= \int_{\alpha_n(\mathbb{Z}_p)} g^{(n),\sharp}\left([x]_n\right)\langle [x]_n, y\rangle \,\mathrm{d}[x]_n \\
&\quad + \int_{\alpha_n(C_1)} g^{(n),\sharp}\left([x]_n\right)\langle [x]_n, y\rangle \,\mathrm{d}[x]_n + \cdots + \int_{\alpha_n(C_{-k+n})} g^{(n),\sharp}\left([x]_n\right)\langle [x]_n, y\rangle \,\mathrm{d}[x]_n \\
&\qquad + \int_{\alpha_n(C_{-k+n+1})} g^{(n),\sharp}\left([x]_n\right)\langle [x]_n, y\rangle \,\mathrm{d}[x]_n \\
&= \frac{p^b-2}{p^b-1}\frac{1}{\mathrm{vol}\left(\alpha_n\left(\mathbb{Z}_p\right)\right)}\int_{\alpha_n(\mathbb{Z}_p)} \langle [x]_n, y\rangle \,\mathrm{d}[x]_n \\
&\quad + \frac{1}{p^b}\frac{1}{\mathrm{vol}\left(\alpha_n\left(C_1\right)\right)}\int_{\alpha_n(C_1)} \langle [x]_n, y\rangle \,\mathrm{d}[x]_n \\
&\qquad + \cdots + \frac{1}{p^{(n-k)b}}\frac{1}{\mathrm{vol}\left(\alpha_n\left(C_{-k+n}\right)\right)}\int_{\alpha_n(C_{-k+n})} \langle [x]_n, y\rangle \,\mathrm{d}[x]_n \\
&\qquad - \frac{1}{p^b(p-1)}\frac{|y|^b}{p^{nb}} \\
&= \frac{p^b-2}{p^b-1} + \frac{1}{p^b} + \cdots + \frac{1}{p^{(n-k)b}} - \frac{1}{p^b(p-1)}\frac{|y|^b}{p^{nb}} \\
&= 1 - \left(\frac{1}{p^{(n-k+1)b}} + \frac{1}{p^{(n-k+2)b}} + \frac{1}{p^{(n-k+3)b}} + \cdots\right) - \frac{1}{p^b(p-1)}\frac{|y|^b}{p^{nb}} \\
&= 1 - \left(\frac{|y|^b}{p^{(n+1)b}} + \frac{|y|^b}{p^{(n+2)b}} + \frac{|y|^b}{p^{(n+3)b}} + \cdots\right) - \frac{1}{p^b(p-1)}\frac{|y|^b}{p^{nb}} \\
&= 1 - \frac{|y|^b}{p^{(n+1)b}}\left(1 + \frac{1}{p^b} + \frac{1}{p^{2b}} + \cdots\right) - \frac{1}{p^b(p-1)}\frac{|y|^b}{p^{nb}} = 1 - \frac{\beta|y|^b}{p^{nb}},
\end{aligned}
$$

where

$$\beta = \left(\frac{1}{p^b-1} + \frac{1}{p^b(p-1)}\right).$$

Take

$$l(n) = p^{nb}\frac{D}{\beta}$$

and fix y in $p^{-k}\mathbb{Z}_p$. If n is greater than k, then $\left(\mathcal{F}_n f_t^{(n),\sharp}\right)(y)$ is well defined and

$$
\begin{aligned}
\left(\mathcal{F}_n f_t^{(n),\sharp}\right)(y) &= \left(\left(\mathcal{F}_n g^{(n),\sharp}\right)(y)\right)^{\lfloor tl(n)\rfloor} \\
&= \left(1 - \frac{\beta|y|^b}{p^{nb}}\right)^{\left\lfloor tp^{nb}\frac{D}{\beta}\right\rfloor} \to \mathrm{e}^{-Dt|y|^b} = (\mathcal{F}f)(t,y)
\end{aligned}
$$

as n tends to infinity. In this case, the limit is the Fourier transform of the solution to (3), the generalized p-adic diffusion equation. Therefore, if the time scaling of the space temporal embedding ι_n is such that the stochastic process $Y^{(n)}$ will converge to the stochastic process Y, this is to say that the measure P on $D([0,\infty)\colon \mathbb{Q}_p)$ is the weak-$*$ limit of the measures P_n on $D([0,\infty)\colon \mathbb{Q}_p)$, then the time scale $l(n)$ must be asymptotically equal to $p^{nb}\frac{D}{\beta}$. To simplify the exposition, simply take l to be such that

$$l(n) = p^{nb}\frac{D}{\beta},$$

although we need only that

$$\frac{l(n)}{p^{nb}} \to \frac{D}{\beta} \quad \text{as} \quad n \to \infty.$$

6. Convergence of the Random Walks

6.1. **Convergence on Cylinder Sets with Restricted Histories.** Let I be a fixed non-empty time interval. Let H_R be the set of all *restricted histories*, histories that are restricted to have routes taking values in the set of finite balls in $\mathbb{Q}_p$ and that have epochs taking values in I. Since the intersection of two balls in $\mathbb{Q}_p$ with non-trivial intersection is again a ball, the subset $C(H_R)$ of $D(I\colon \mathbb{Q}_p)$ is a π-system. Moreover, $C(H_R)$ generates the same σ-algebra as $C(H)$. To compress notation, denote for each integer i

$$S_i = S_i(0) \quad \text{and} \quad B_i = B_i(0).$$

For each fixed positive real number t, define the function $g_n(t,\cdot)$ on $p^{-n}\mathbb{Z}_p$ by

$$g_n(t,y) = \left(1 - \frac{\beta|y|^b}{p^{nb}}\right)^{\lfloor tl(n)\rfloor}$$

and denote

$$g_{t,n}(i) = g_n(t,y) \quad \text{where} \quad y \in S_i.$$

Henceforth, denote $[x]_n$ by $[x]$ and the context will specify the meaning of the symbol.

Proposition 5. *The function $f_t^{(n),\sharp}$ is the inverse Fourier transform of $g_n(t,\cdot)$ and*

$$f_t^{(n),\sharp}([x]) = p^n g_{t,n}(n)\mathbb{1}_{B_{-n}}([x]) + \sum_{k\leq n-1} p^k\left(g_{t,n}(k) - g_{t,n}(k+1)\right)\mathbb{1}_{B_{-k}}([x]).$$

Proof. Since $g_n(t,\cdot)$ is the Fourier transform of $f_t^{(n),\sharp}$, take the inverse Fourier transform of $g_n(t,\cdot)$ to obtain the equality

$$\begin{aligned} f_t^{(n),\sharp}([x]) &= \left(\mathcal{F}_n^{-1} g_n(t,\cdot)\right)([x]) \\ &= \sum_{k\leq n} \int_{S_k} \langle [x], y\rangle g_n(t,y)\,\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\le n} g_{t,n}(k) \int_{S_k} \langle [x], y\rangle \,\mathrm{d}y \\
&= \sum_{k\le n} g_{t,n}(k) \left(\int_{B_k} \langle [x], y\rangle \,\mathrm{d}y - \int_{B_{k-1}} \langle [x], y\rangle \,\mathrm{d}y \right) \\
&= g_{t,n}(n) \int_{B_n} \langle [x], y\rangle \,\mathrm{d}y + \sum_{k\le n-1} \left(g_{t,n}(k) - g_{t,n}(k+1)\right) \int_{B_k} \langle [x], y\rangle \,\mathrm{d}y \\
&= p^n g_{t,n}(n) \mathbb{1}_{B_{-n}}([x]) + \sum_{k\le n-1} p^k \left(g_{t,n}(k) - g_{t,n}(k+1)\right) \mathbb{1}_{B_{-k}}([x]). \qquad \square
\end{aligned}
$$

Minor modifications of the arguments in [18] prove the equality

$$
f(t,x) = \sum_{k\in\mathbb{Z}} p^k \left(\mathrm{e}^{-Dtp^{kb}} - \mathrm{e}^{-Dtp^{(k+1)b}} \right) \mathbb{1}_{B_{-k}}(x),
$$

where $f(t,x)$ is the solution of (3).

Proposition 6. *If h is a history in H_R, then*

$$
P_n(C(h)) \to P(C(h)).
$$

Proof. Suppose that h is in H_R, that length(h) is ℓ and that the route of h, $U(h)$, and the epoch of h, $e(h)$, are given by

$$
U(h) = \{V_0, \ldots, V_\ell\} \quad \text{and} \quad e(h) = (t_1, \ldots, t_\ell).
$$

For any natural number n, if V_0 does not contain zero, then $P(C(h))$ and $P_n(C(h))$ are zero. Assume, therefore, that V_0 contains only the point 0. For any ball $B_r(x)$ with radius p^r centered at x in $\mathbb{Q}_p$, if $-n$ is less than r, then

$$
B_r(x) = \bigcup_{y\in j_n(B_r([x]))} B_{-n}(y)
$$

where the given union is a disjoint union. Therefore, for sufficiently large n, there are for each i in $\{1, \ldots, \ell\}$ natural numbers n_i and elements β_{i,k_i} of $j_n(G_n)$ with k_i in $\{1, \ldots, n_i\}$ such that

$$
V_i = \bigcup_{k_i\in\{1,\ldots,n_i\}} B_{-n}(\beta_{i,k_i}).
$$

Notice that, in this case,

$$\begin{aligned}
P_n(C(h)) &= P_n\left(\left\{\omega \in D(I\colon \mathbb{Q}_p)\colon \omega(t_1)=0, \omega(t_1)\in \bigcup_{k_1=1}^{n_1} B_{-n}(\beta_{1,k_1}),\right.\right.\\
&\qquad\qquad \left.\left. \dots, \omega(t_\ell)\in \bigcup_{k_\ell=1}^{n_\ell} B_{-n}(\beta_{\ell,k_\ell})\right\}\right)\\
&= P_n\left(\left\{\omega \in D(I\colon \mathbb{Q}_p)\colon \omega(t_1)=0, \omega(t_1)\in \bigcup_{k_1=1}^{n_1} \{\beta_{1,k_1}\},\right.\right. \qquad (5)\\
&\qquad\qquad \left.\left. \dots, \omega(t_\ell)\in \bigcup_{k_\ell=1}^{n_\ell} \{\beta_{\ell,k_\ell}\}\right\}\right)\\
&= P^{(n),\sharp}\left(\left\{\omega \in D(I\colon G_n)\colon \omega(t_1)=[0], \omega(t_1)\in \bigcup_{k_1=1}^{n_1} \{[\beta_{1,k_1}]\},\right.\right.\\
&\qquad\qquad \left.\left. \dots, \omega(t_\ell)\in \bigcup_{k_\ell=1}^{n_\ell} \{[\beta_{\ell,k_\ell}]\}\right\}\right)\\
&= \int_{[V_1]}\cdots\int_{[V_\ell]} f_{t_1}^{(n),\sharp}([x_1]) f_{t_2-t_1}^{(n),\sharp}([x_2]-[x_1])\cdots f_{t_\ell-t_{\ell-1}}^{(n),\sharp}([x_\ell]-[x_{\ell-1}])\,\mathrm{d}[x_1]\cdots\mathrm{d}[x_\ell]\\
&= \int_{V_1}\cdots\int_{V_\ell} f_{t_1}^{(n),\sharp}([x_1]) f_{t_2-t_1}^{(n),\sharp}([x_2]-[x_1])\cdots f_{t_\ell-t_{\ell-1}}^{(n),\sharp}([x_\ell]-[x_{\ell-1}])\,\mathrm{d}x_1\cdots\mathrm{d}x_\ell\\
&= \int_{V_1}\cdots\int_{V_\ell} f_{t_1}^{(n),\sharp}([x_1]) f_{t_2-t_1}^{(n),\sharp}([x_2-x_1])\cdots f_{t_\ell-t_{\ell-1}}^{(n),\sharp}([x_\ell-x_{\ell-1}])\,\mathrm{d}x_1\cdots\mathrm{d}x_\ell.
\end{aligned}$$

Note that equality (5) holds since Proposition 3 implies that paths taking values in $j_n(G_n)$ have full measure with respect to P_n. It is straightforward to see that for any fixed T_1 larger than zero and T_2 larger than T_1 and for any fixed ball B, $\left|f(t,x) - f_t^{(n),\sharp}([x])\right|$ tends to zero uniformly as t varies in $[T_1, T_2]$ and x varies in B. The equality

$$\begin{aligned}
P(C(h)) &= P(\{\omega \in D(I\colon \mathbb{Q}_p)\colon \omega(t_1)=0, \omega(t_1)\in V_1, \dots, \omega(t_\ell)\in V_\ell\})\\
&= \int_{V_1}\cdots\int_{V_\ell} f(t_1,x_1) f(t_2-t_1, x_2-x_1)\cdots f(t_\ell - t_{\ell-1}, x_\ell - x_{\ell-1})\,\mathrm{d}x_1\cdots\mathrm{d}x_\ell
\end{aligned}$$

implies that

$$\begin{aligned}
&|P(C(h)) - P_n(C(h))| \\
&\quad = \left| \int_{V_1} \cdots \int_{V_\ell} f(t_1, x_1) f(t_2 - t_1, x_2 - x_1) \cdots f(t_\ell - t_{\ell-1}, x_\ell - x_{\ell-1})\, \mathrm{d}x_1 \cdots \mathrm{d}x_\ell \right. \\
&\qquad\qquad - \int_{V_1} \cdots \int_{V_\ell} f^{(n),\sharp}_{t_1}([x_1]) f^{(n),\sharp}_{t_2-t_1}([x_2 - x_1]) \\
&\qquad\qquad\qquad\qquad \left. \cdots f^{(n),\sharp}_{t_\ell - t_{\ell-1}}([x_\ell - x_{\ell-1}])\, \mathrm{d}x_1 \cdots \mathrm{d}x_\ell \right| \\
&\quad \le \int_{V_1} \cdots \int_{V_\ell} \Big| f(t_1, x_1) f(t_2 - t_1, x_2 - x_1) \cdots f(t_\ell - t_{\ell-1}, x_\ell - x_{\ell-1}) \\
&\qquad\qquad - f^{(n),\sharp}_{t_1}([x_1]) f^{(n),\sharp}_{t_2-t_1}([x_2 - x_1]) \cdots f^{(n),\sharp}_{t_\ell - t_{\ell-1}}([x_\ell - x_{\ell-1}]) \Big|\, \mathrm{d}x_1 \cdots \mathrm{d}x_\ell
\end{aligned}$$

and the righthand term of the ultimate inequality tends to zero uniformly on $V_1 \times \cdots \times V_\ell$ as n tends to infinity. ☐

6.2. **Moment Estimates.** The moment estimates for $\big|Y^{(n)}_t\big|$ obtained in Lemma 1 lacked uniformity in n. However, it is possible to achieve uniformity in these estimates.

Proposition 7. *For each positive real number T and positive real number r less than b, there is a constant C such that for any natural number n and any t in $[0, T]$, the real valued random variable $\big|Y^{(n)}_t\big|$ satisfies the moment estimate*

$$\mathrm{E}\Big[\big|Y^{(n)}_t\big|^r\Big] < Ct^{\frac{r}{b}}.$$

Proof. For a fixed t in $[0, T]$, define the sequence (t_n) by

$$t_n l(n) = \lfloor t l(n) \rfloor.$$

For each t and each natural number n, t_n is bounded above by t and

$$t_n l(n) < 1 \quad \text{implies that} \quad \mathrm{E}\Big[\big|Y^{(n)}_t\big|^r\Big] = 0.$$

Therefore, assume in the calculations below that $t_n l(n)$ is at least equal to one. The equality of $\mathrm{E}\Big[\big|Y^{(n)}_t\big|^r\Big]$ and $\mathrm{E}\Big[\big|Y^{(n),\sharp}_t\big|^r\Big]$ implies that

$$\begin{aligned}
\mathrm{E}\Big[\big|Y^{(n)}_t\big|^r\Big] &= \mathrm{E}\Big[\big|Y^{(n),\sharp}_t\big|^r\Big] \\
&= \int_{G_n} |[x]|^r f^{(n),\sharp}_t([x])\, \mathrm{d}[x]
\end{aligned}$$

$$
\begin{aligned}
&= \int_{G_n} |[x]|^r \Bigg\{ p^n g_{t,n}(n) \mathbb{1}_{B_{-n}}([x]) \\
&\qquad + \sum_{k \le n-1} p^k \left(g_{t,n}(k) - g_{t,n}(k+1) \right) \mathbb{1}_{B_{-k}}([x]) \Bigg\} \, \mathrm{d}[x] \\
&= p^n g_{t,n}(n) \int_{G_n} |[x]|^r \mathbb{1}_{B_{-n}}(x) \, \mathrm{d}[x] \\
&\qquad + \int_{G_n} |[x]|^r \left\{ \sum_{k \le n-1} p^k \left(g_{t,n}(k) - g_{t,n}(k+1) \right) \mathbb{1}_{B_{-k}}([x]) \right\} \mathrm{d}[x] \\
&= \sum_{k \le n-1} p^k \left(g_{t,n}(k) - g_{t,n}(k+1) \right) \int_{G_n} |[x]|^r \mathbb{1}_{B_{-k}}([x]) \, \mathrm{d}[x]. \qquad (6)
\end{aligned}
$$

Together with (6), the bound

$$
\int_{G_n} |[x]|^r \mathbb{1}_{B_{-k}}([x]) \, \mathrm{d}[x] = p^{-kr} \int_{G_n} \mathbb{1}_{B_{-k}}([x]) \, \mathrm{d}[x] \le p^{-kr} p^{-k}
$$

implies that

$$
\mathrm{E}\left[\left| Y_t^{(n)} \right|^r \right] \le \sum_{k \le n-1} p^{-kr} \left| g_{t,n}(k) - g_{t,n}(k+1) \right| . \qquad (7)
$$

Denote by E_k the quantity

$$
E_k = g_{t,n}(k) - g_{t,n}(k+1)
$$

and re-write (7) as

$$
\begin{aligned}
\mathrm{E}\left[\left| Y_t^{(n)} \right|^r \right] &\le p^{r(-n+1)} \left| E_{n-1} \right| + p^{r(-n+2)} \left| E_{n-2} \right| \\
&\qquad + \cdots + p^{-r} \left| E_1 \right| + p^{0r} \left| E_0 \right| + p^{r} \left| E_{-1} \right| + p^{2r} \left| E_{-2} \right| + \cdots .
\end{aligned}
$$

Below, we will work with the quantity $\left(1 - \frac{\beta p^{kb}}{p^{nb}} \right)$. If p is not two then this quantity is strictly larger than zero and, in fact, is bounded away from zero by a bound that is independent of k and n. If p is equal to two, then when k equals n and b is in $\left(1, \log_2\left(3+\sqrt{5}\right) - 1\right]$ this quantity is non-positive and equal to zero precisely at the right endpoint of the interval.

In the case when $k+1$ is less than n, or when p is greater than two, or when p is two but b is larger than $\log_2\left(3+\sqrt{5}\right)-1$, the Fundamental Theorem of Calculus implies that

$$\begin{aligned}
E_k &= \left(1-\frac{\beta p^{kb}}{p^{nb}}\right)^{\lfloor tl(n)\rfloor} - \left(1-\frac{\beta p^{(k+1)b}}{p^{nb}}\right)^{\lfloor tl(n)\rfloor} \\
&= -\int_{p^{kb}}^{p^{(k+1)b}} \frac{\mathrm{d}}{\mathrm{d}s}\left(1-\frac{\beta s}{p^{nb}}\right)^{\lfloor tl(n)\rfloor} \mathrm{d}s \\
&= \int_{p^{kb}}^{p^{(k+1)b}} \frac{\beta}{p^{nb}}\lfloor tl(n)\rfloor \left(1-\frac{\beta s}{p^{nb}}\right)^{\lfloor tl(n)\rfloor-1} \mathrm{d}s \\
&\leq \left(p^{(k+1)b}-p^{kb}\right)\frac{\beta}{p^{nb}}\lfloor tl(n)\rfloor\left(1-\frac{\beta p^{kb}}{p^{nb}}\right)^{\lfloor tl(n)\rfloor-1} \\
&= \left(p^{(k+1)b}-p^{kb}\right)\frac{\beta}{p^{nb}}t_n l(n)\left(1-\frac{\beta p^{kb}}{p^{nb}}\right)^{\lfloor tl(n)\rfloor-1} \\
&\leq \left(p^{(k+1)b}-p^{kb}\right)\frac{\beta}{p^{nb}}t_n l(n)\left(1-\frac{\beta p^{kb}}{p^{nb}}\right)^{-1}\left(1-\frac{\beta p^{kb}}{p^{nb}}\right)^{\lfloor tl(n)\rfloor} \\
&= \left(p^{b}-1\right)\beta\frac{l(n)}{p^{nb}}\left(1-\frac{\beta p^{kb}}{p^{nb}}\right)^{-1} t_n\left\{p^{kb}\left(1-\frac{\beta p^{kb}}{p^{nb}}\right)^{\lfloor tl(n)\rfloor}\right\} < C_1 t_n p^{kb} G_k,
\end{aligned}$$

where

$$G_k = \left(1-\frac{\beta p^{kb}}{p^{nb}}\right)^{\lfloor tl(n)\rfloor}$$

and C_1 is a constant, independent of k and n with

$$C_1 > \left(p^{b}-1\right)\beta\frac{l(n)}{p^{nb}}\left(1-\frac{\beta p^{kb}}{p^{nb}}\right)^{-1}.$$

The positivity of the E_k in the current setting implies the estimate

$$0 < E_k < C_1 t_n p^{kb} G_k. \tag{8}$$

The case when $k+1$ is equal to n requires additional attention when p is two, though we treat all p in this case to simplify the exposition. The assumption that $\lfloor tl(n)\rfloor$ is at least one, implies that there is a natural number m such that $\lfloor tl(n)\rfloor$ is equal to m, that is

$$t_n = \frac{m}{l(n)}.$$

There is a real number a in $(0,1)$ such that

$$\begin{aligned}|E_{n-1}| &\le \left|1-\frac{\beta}{p^b}\right|^{\lfloor tl(n)\rfloor} + |1-\beta|^{\lfloor tl(n)\rfloor} \\ &\le 2a^{\lfloor tl(n)\rfloor} = 2a^m.\end{aligned}$$

Therefore,

$$\begin{aligned}p^{r(-n+1)}\,|E_{n-1}| &\le 2p^{-rn}p^r a^m \\ &= 2\left(\left(p^b\right)^{\frac{1}{b}}\right)^{-rn} p^r a^m \\ &= 2\left(p^{-nb}\right)^{\frac{r}{b}} p^r a^m.\end{aligned}$$

There is a constant L such that

$$2p^r a^m < Lm^{\frac{r}{b}}\left(\frac{\beta}{D}\right)^{\frac{r}{b}},$$

and so

$$\begin{aligned}p^{r(-n+1)}\,|E_{n-1}| &\le \left(p^{-nb}\right)^{\frac{r}{b}} Lm^{\frac{r}{b}}\left(\frac{\beta}{D}\right)^{\frac{r}{b}} \\ &= L\left(\frac{m}{p^{nb}\frac{D}{\beta}}\right)^{\frac{r}{b}} = Lt_n^{\frac{r}{b}}. \qquad (9)\end{aligned}$$

Combine (8) and (9) to obtain the inequality

$$\begin{aligned}\mathrm{E}\left[\left|Y_t^{(n)}\right|^r\right] &\le Lt_n^{\frac{r}{b}} + C_1 t_n\left\{p^{r(-n+2)}p^{b(n-2)}G_{n-2} + p^{r(-n+3)}p^{b(n-3)}G_{n-3}\right. \\ &\qquad \left. + \cdots + p^{-r}p^{1b}G_1 + G_0 + p^{1r}p^{-1b}G_{-1} + p^{2r}p^{-2b}G_{-2} + \cdots\right\} \\ &= Lt_n^{\frac{r}{b}} + C_1 t_n\left\{p^{(n-2)(b-r)}G_{n-2} + p^{(n-3)(b-r)}G_{n-3}\right. \\ &\qquad \left. + \cdots + p^{1(b-r)}G_1 + p^{0(b-r)}G_0 + p^{-1(b-r)}G_{-1} + p^{-2(b-r)}G_{-2} + \cdots\right\}.\end{aligned}$$

Define

$$I = p^{(n-2)(b-r)}G_{n-2} + p^{(n-3)(b-r)}G_{n-3} + \cdots + p^{1(b-r)}G_1 + p^{0(b-r)}G_0$$

and

$$II = p^{-1(b-r)}G_{-1} + p^{-2(b-r)}G_{-2} + \cdots.$$

The inequality

$$|G_k| \le 1$$

implies that

$$II \le C_2 = \frac{1}{p^{b-r}-1}.$$

Since βp^{kb} is less than p^{nb},

$$G_k = \left(1 - \frac{\beta p^{kb}}{p^{nb}}\right)^{\lfloor tl(n) \rfloor} = \left(1 - \frac{\beta p^{kb}}{p^{nb}}\right)^{t_n l(n)} \le \mathrm{e}^{-Dt_n p^{kb}},$$

and so

$$\begin{aligned}
\mathrm{E}\Big[\big|Y_t^{(n)}\big|^r\Big] &< Lt_n^{\frac{r}{b}} + C_1 t_n \Big\{p^{(n-2)(b-r)}\mathrm{e}^{-Dt_n p^{b(n-2)}} \\
&\qquad + p^{(n-3)(b-r)}\mathrm{e}^{-Dt_n p^{b(n-3)}} + \cdots + p^{1(b-r)}\mathrm{e}^{-Dt_n p^{1b}} + p^{0(b-r)}\mathrm{e}^{-Dt_n p^{b0}} + C_2\Big\} \\
&< Lt_n^{\frac{r}{b}} + C_1 t_n \Big\{p^{(n-2)(b-r)}\mathrm{e}^{-Dt_n p^{b(n-2)}} \\
&\qquad + p^{(n-3)(b-r)}\mathrm{e}^{-Dt_n p^{b(n-3)}} + \cdots + p^{1(b-r)}\mathrm{e}^{-Dt_n p^{1b}} + 1 + C_2\Big\} \\
&= Lt_n^{\frac{r}{b}} + C_1 t_n \Big\{p^{(n-2)(b-r)}\mathrm{e}^{-Dt_n p^{b(n-2)}} + p^{(n-3)(b-r)}\mathrm{e}^{-Dt_n p^{b(n-3)}} \\
&\qquad + \cdots + p^{1(b-r)}\mathrm{e}^{-Dt_n p^{1b}}\Big\} + C_1 t_n \left(1 + C_2\right).
\end{aligned}$$

To simplify the expressions, write

$$s_n = Dt_n$$

and estimate the sum

$$\begin{aligned}
III &= p^{(n-2)(b-r)}\mathrm{e}^{-Dt_n p^{b(n-2)}} + p^{(n-3)(b-r)}\mathrm{e}^{-Dt_n p^{b(n-3)}} + \cdots + p^{1(b-r)}\mathrm{e}^{-Dt_n p^{1b}} \\
&= p^{(n-2)(b-r)}\mathrm{e}^{-s_n p^{b(n-2)}} + p^{(n-3)(b-r)}\mathrm{e}^{-s_n p^{b(n-3)}} + \cdots + p^{1(b-r)}\mathrm{e}^{-s_n p^{1b}} \\
&= \frac{1}{s_n^{\frac{b-r}{b}}} \Bigg\{ \left(s_n^{\frac{1}{(n-2)b}} p\right)^{(n-2)(b-r)} \mathrm{e}^{-\left(s_n^{\frac{1}{(n-2)b}} p\right)^{(n-2)b}} \\
&\qquad + \left(s_n^{\frac{1}{(n-3)b}} p\right)^{(n-3)(b-r)} \mathrm{e}^{-\left(s_n^{\frac{1}{(n-3)b}} p\right)^{(n-3)b}} + \cdots + \left(s_n^{\frac{1}{1b}} p\right)^{1(b-r)} \mathrm{e}^{-\left(s_n^{\frac{1}{1b}} p\right)^{1b}} \Bigg\}.
\end{aligned}$$

If the term in the braces is uniformly bounded in n and t by a constant C_3, then

$$\begin{aligned}
\mathrm{E}\Big[\big|Y_t^{(n)}\big|^r\Big] &< Lt_n^{\frac{r}{b}} + C_1 t_n \left\{ \frac{1}{s_n^{\frac{b-r}{b}}} C_3 + (1 + C_2) \right\} \\
&= Lt_n^{\frac{r}{b}} + C_1 C_3 t_n s_n^{\frac{r}{b}-1} + (1 + C_2) C_1 t_n \\
&= Lt_n^{\frac{r}{b}} + C_1 C_3 t_n (Dt_n)^{\frac{r}{b}-1} + (1 + C_2) C_1 t_n \\
&= Lt_n^{\frac{r}{b}} + C_1 C_3 D^{\frac{r}{b}-1} t_n^{\frac{r}{b}} + (1 + C_2) C_1 t_n < Kt_n^{\frac{r}{b}} < Kt^{\frac{r}{b}},
\end{aligned}$$

where K is independent of t and n as long as t varies in the interval $[0, T]$ for fixed T.

To prove the proposition, it now suffices now to show that

$$\begin{aligned} J_n(t) &= \left(s_n^{\frac{1}{(n-2)b}}p\right)^{(n-2)(b-r)} \mathrm{e}^{-\left(s_n^{\frac{1}{(n-2)b}}p\right)^{(n-2)b}} \\ &\quad + \left(s_n^{\frac{1}{(n-3)b}}p\right)^{(n-3)(b-r)} \mathrm{e}^{-\left(s_n^{\frac{1}{(n-3)b}}p\right)^{(n-3)b}} + \cdots + \left(s_n^{\frac{1}{1b}}p\right)^{1(b-r)} \mathrm{e}^{-\left(s_n^{\frac{1}{1b}}p\right)^{1b}} \\ &= s_n^{\frac{b-r}{b}}p^{(n-2)(b-r)}\mathrm{e}^{-s_np^{(n-2)b}} + s_n^{\frac{b-r}{b}}p^{(n-3)(b-r)}\mathrm{e}^{-s_np^{(n-3)b}} + \cdots + s_n^{\frac{b-r}{b}}p^{1(b-r)}\mathrm{e}^{-s_np^{b}} \end{aligned}$$

is bounded by a constant. Note that

$$t_nl(n) = t_n\frac{D}{\beta}p^{nb}$$

is a natural number or is equal to zero if $tl(n)$ is less than one. In the latter case, the expected value is zero. Therefore, assume that there is a natural number m such that

$$t_n = \frac{m\beta}{D}p^{-nb}.$$

For some constant C in $\left[1, p^b\right]$ and for some l which is either a natural number or zero, write m as

$$m = Cp^{lb} \quad \text{so that} \quad t_n = \frac{C\beta}{D}p^{(l-n)b}.$$

This is to say that

$$s_n = C\beta p^{(l-n)^b}.$$

Therefore,

$$\begin{aligned} J_n(t) &= s_n^{\frac{b-r}{b}}p^{(n-2)(b-r)}\mathrm{e}^{-s_np^{(n-2)b}} \\ &\qquad + s_n^{\frac{b-r}{b}}p^{(n-3)(b-r)}\mathrm{e}^{-s_np^{(n-3)b}} + \cdots + s_n^{\frac{b-r}{b}}p^{1(b-r)}\mathrm{e}^{-s_np^{b}} \\ &= \left(C\beta p^{(l-n)b}\right)^{\frac{b-r}{b}}p^{(n-2)(b-r)}\mathrm{e}^{-\left(C\beta p^{(l-n)b}\right)p^{(n-2)b}} \\ &\qquad + \left(C\beta p^{(l-n)b}\right)^{\frac{b-r}{b}}p^{(n-3)(b-r)}\mathrm{e}^{-\left(C\beta p^{(l-n)b}\right)p^{(n-3)b}} \\ &\qquad\quad + \cdots + \left(C\beta p^{(l-n)b}\right)^{\frac{b-r}{b}}p^{1(b-r)}\mathrm{e}^{-\left(C\beta p^{(l-n)b}\right)p^{b}} \\ &= (C\beta)^{\frac{b-r}{b}}\left\{p^{(l-2)(b-r)}\mathrm{e}^{-\left(C\beta p^{(l-2)b}\right)}\right. \\ &\qquad + p^{(l-3)(b-r)}\mathrm{e}^{-\left(C\beta p^{(l-3)b}\right)} + \cdots + p^{0(b-r)}\mathrm{e}^{-\left(C\beta p^{0b}\right)} \\ &\qquad\quad \left. + p^{-(b-r)}\mathrm{e}^{-\left(C\beta p^{-b}\right)} + \cdots + p^{(l+1-n)(b-r)}\mathrm{e}^{-\left(C\beta p^{(l+1-n)b}\right)}\right\}. \end{aligned}$$

If x is positive, then

$$\mathrm{e}^{-x} \leq \min\left\{1, \frac{1}{x}\right\}$$

and so

$$\begin{aligned} J_n(t) &\leq (C\beta)^{\frac{b-r}{b}} \left\{ \frac{p^{(l-2)(b-r)}}{(C\beta p^{(l-2)b})} + \frac{p^{(l-3)(b-r)}}{(C\beta p^{(l-3)b})} + \cdots + \frac{p^{0(b-r)}}{(C\beta p^{0b})} \right. \\ &\qquad \left. + p^{-(b-r)} + \cdots + p^{(l+1-n)(b-r)} \right\} \\ &\leq (C\beta)^{\frac{b-r}{b}} \left\{ \frac{1}{C\beta} \left((p^r)^{-(l-2)} + (p^r)^{-(l-3)} + \cdots + (p^r)^0 \right) \right. \\ &\qquad \left. + p^{-(b-r)} + \cdots + p^{(l+1-n)(b-r)} \right\} \\ &< \left(\frac{1}{C\beta} \right)^{\frac{r}{b}} \left\{ \frac{1}{1 - \frac{1}{p^r}} \right\} + (C\beta)^{\frac{b-r}{b}} \left\{ \frac{1}{1 - \frac{1}{p^{b-r}}} \right\}. \end{aligned}$$

□

6.3. **Convergence of Measures.** Now that we have made uniform moment estimates for the measures P_n and have proved the convergence of the distributions on a π-system, we are ready to state and prove our main result. If T is a positive real number then the set $D([0,T]\colon \mathbb{Q}_p)$ has a canonical mapping into the subset of $D([0,\infty)\colon \mathbb{Q}_p)$ where no restrictions are placed on paths after time T. Consistency of the measure under projection allows any probability measure on $D([0,\infty)\colon \mathbb{Q}_p)$ to be viewed as a probability measure on $D([0,T]\colon \mathbb{Q}_p)$. For the next theorem, view the measures previously developed to be measures on $D([0,T]\colon \mathbb{Q}_p)$.

Theorem 1. *Suppose that (ι_n) is a sequence of spatiotemporal embeddings of the primitive discrete time process S with the property that*

$$\delta(n) = p^{-n} \quad \text{and} \quad \tau(n) = \frac{1}{p^{nb\frac{D}{\beta}}}.$$

Fix a positive real number T. Let P be the measure on $D([0,T]\colon \mathbb{Q}_p)$ that is associated to the process Y. For each n in $\mathbb{N}_0$, let P_n be the measure on $D([0,T]\colon \mathbb{Q}_p)$ associated to the process

$$Y^n = \iota_n S.$$

The sequence of measures (P_n) converge in the weak-$*$ topology on $D([0,T]\colon \mathbb{Q}_p)$ to the measure P.

Proof. Proposition 6 implies that there is a π-system $\mathcal{A}$ that generates the same σ-algebra as that which $C(H)$ generates so that for each A in $\mathcal{A}$,

$$P_n(A) \to P(A).$$

By Proposition 7, there is a constant C that is independent of t and n so that

$$\mathrm{E}\left[\left|Y_t^{(n)}\right|^r\right] < Ct^{\frac{r}{b}}.$$

Therefore, if (t_1, t_2, t_3) is an epoch in $[0, T]$, then the independence of the increments of Y implies that

$$\begin{aligned}\mathrm{E}\left[\left|Y_{t_3}^{(n)} - Y_{t_2}^{(n)}\right|^r\left|Y_{t_2}^{(n)} - Y_{t_1}^{(n)}\right|^r\right] &= \mathrm{E}\left[\left|Y_{t_3}^{(n)} - Y_{t_2}^{(n)}\right|^r\right]\mathrm{E}\left[\left|Y_{t_2}^{(n)} - Y_{t_1}^{(n)}\right|^r\right]\\ &\le C(t_3-t_2)^{\frac{r}{b}}C(t_2-t_1)^{\frac{r}{b}} \le C^2(t_3-t_1)^{\frac{2r}{b}}.\end{aligned}$$

Taking r to be in $\left(\frac{b}{2}, b\right)$ verifies that Y satisfies Centsov's criterion with constant C independent of the epoch (t_1, t_2, t_3) and uniform in n. This proves the theorem [9].

□

Notice the similarity of the scaling in the p-adic and real settings. In the real setting, if δ is a distance scale and τ is a time scale then a sequence of spatiotemporal embeddings of the primitive random walk on $\mathbb{Z}$ converges to a Brownian motion when the scales have the property that for some constant C,

$$\frac{\delta(n)^2}{\tau(n)} \to C.$$

In the p-adic setting with b strictly greater than one and the diffusion equation given by (3), we have shown that if a sequence of spatiotemporal embeddings of the primitive random walk on $\mathbb{Q}_p/\mathbb{Z}_p$ satisfies the property that

$$\delta(n)^b = \frac{D}{\beta}\tau(n),$$

then the sequence of spatiotemporal embeddings of the primitive random walk converges to a p-adic Brownian motion. In fact, with minor alterations to the proof, we can more generally prove that the same convergence holds as long as the sequence of spatiotemporal embeddings satisfy the weaker condition that

$$\frac{\delta(n)^b}{\tau(n)} \to \frac{D}{\beta}.$$

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