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Stochastic Analysis of

England Grain Prices 1750-1850

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Abstract

We study the prices of grain in England 1750-1850, with special attention to the *Tamboro* volcano eruption time period 1815-1820. Two different types of time series are constructed and compared with the historical data. We find that the 1750-1789 time series, before the 1790-1814 Napoleon war, is much like the time series 1821-1850. However, the time series during the 1815-1820 *Tamboro* volcano eruption period is clearly very different from these two time series. Prices appear significantly outside the confidence 3 sigma level, where sigma is measured for the 1750-1789 period. The prices during the volcano eruption period 1815-1820 are significantly larger than the prices during the periods 1750-1789 and 1821-1850. Our general conclusion is that the historical reports about increasing food prices due to the volcano eruptions is supported by our statistical analysis of the historical data of grain prices. The time series during the Napoleon war show very high prices and extreme fluctuations and are not analyzed in detail.

Keywords: Aerosols, volcanic eruption, time series, stochastic theory, Markov process, Ito integrals, Stratonovich integrals, differential equations, probability densities, grain prices

1 Introduction

1.1 The impact of stratospheric aerosol on climate.
Factual information about changes of climatic conditions in the past is available roughly for the last 600 million years (Phanerozoic). It has been found that the mean air temperature varied mainly due to changes in the CO2 concentration. However, solar radiation and the albedo of the earth are also operative. A doubling in the CO2 concentration gives a temperature increase of around 3 C. 1% increase in the solar radiation increases the temperature around 1-2 C. 1% increase in the albedo decreases the temperature of the earth by 2C. The probability of the long existence of the biosphere of around 4 million years on earth is indeed very low. The biosphere has a narrow temperature range within which most organisms can survive. At any time in the entire history of the earth biosphere, it could have been destroyed as a result of changes in two ongoing independent processes; the growth of the sun's luminosity or the gradual retardation of the earth's degassing from the depth of the earth due to decreasing mass of long lived radioactive isotopes of some of the elements heating this depth.

The discovery that the climate could be catastrophically changed due to enhanced anthropogenic CO2 emission is one of the most important results of the scientific revolution. Indeed, the physical mechanisms of the present warming are identical to the mechanisms that evoked numerous warming and cooling periods in the geological past, which were mainly produced by an increase or decrease in the atmospheric carbon dioxide.

The conclusion that the climate could be catastrophically changed after large scale nuclear warfare was first recognized in the Soviet Union in 1960 (Budyko et al. 1988). The necessity of eliminating the possibility of an uncontrolled nuclear arms race and a large-scale nuclear conflict is an important issue of our time. The major cause of the global climatic change from nuclear warfare is a comparatively short-term but sharp increase in the number of atmospheric aerosol particles that change the albedo of the earth. The accurate estimation of the aerosol amount produced by a large scale nuclear exchange is very difficult.

The fundamental principles for the initiation of an aerosol catastrophe and a possible anthropogenic climate catastrophe are close to each other. However, the accuracy of the calculations is limited. One could also speculate whether stimulated enhanced aerosol concentration is a logistically feasible method of achieving rapid cooling if reduced CO2 emission is not possible. This is a risky business and reduced CO2 emission should therefore be the preferred method whenever political feasible.

There is sufficient empirical information on aerosol produced by explosive volcanic eruption.
The question of aerosol impact on climate in the geological past is important to confirm the reality of possible aerosol climatic changes. The stratospheric aerosol mass grows considerably after explosive volcanic eruptions. A considerable amount of sulfur containing gases enters the atmosphere, producing sulfuric acid droplets and silicate particles attenuating the influx of short

wave radiation to the troposphere. Direct observation of stratospheric aerosol composition has shown that the aerosol consists basically of droplets of water solution of concentrated sulfuric acid. Kimball (1918) was among the first authors to establish that after explosive volcanic eruption, solar radiation to the earth's surface decreased drastically. It has been demonstrated that in individual months the atmospheric aerosol attenuated direct radiation by more than 20 percent (Budyko et al. 1988). If the source of stratospheric aerosol is localized in extra tropic latitudes, aerosol spreads comparatively rapidly over the corresponding hemisphere and more slowly over the other hemisphere. If close to the equator, it is distributed rapidly over both hemispheres. Aerosol in the troposphere (roughly below 10000 meters) is removed during a time period of 2-3 weeks due to rainout. However, particles of stratospheric aerosol only gradually fall out due to both the force of gravity and large scale air motion transferring them to the troposphere where they fall out. The average residence time of stratospheric aerosol particles is 1-2 years (i.e. the time during which their numbers decreases by $e \approx 2.71$ times).

Since the initiating of a world metrological network the eruption of the Krakatoa in 1883 is probably the most well known volcanic explosion. Approximately 20 km^3 of pumice and ash were introduced into the atmosphere. After this eruption unusually bright sunsets were observed all over the world. This is explained by the effect of a sharp increase in stratospheric aerosol mass.

Far greater explosive eruptions occurred in earlier times. The eruption of Tambora 1815 (11. April) was large, and as a result 150-180 km^3 of pumice ash were introduced into the atmosphere. It was rated 7 on the Volcanic Explosive Index[1]. The activity ceased on 15 July 1815. Most deaths from the eruption were from starvation and disease, as the eruptive fallout ruined agricultural productivity in the local region. Due to the drastic decrease in crop yield in number of regions far from the volcano many died of starvation (Stommel and Stommel 1979, 1983). The death toll was at least 71,000 people, of whom 11,000–12,000 were killed directly by the eruption. The eruption may have created global climate anomalies and 1816 became known as the "Year without Summer" because of the North American and European weather, with snow in June and July and frosts through August. The climatic aberrations of 1816 had the greatest effect on the northeastern United States, the Canadian Maritimes, Newfoundland, and Northern Europe. Typically, the late spring and summer of the northeastern U.S. and southeastern Canada are relatively stable: temperatures (average of both day and night) average about 20–25 °C and rarely fall below 5 °C. Summer snow is an extreme rarity. In the spring and summer of 1816, a persistent dry fog was observed in the northeastern United States. The fog reddened and dimmed the sunlight, such that sunspots were visible to the naked eye.

For the past several millennia, data on large explosive eruptions can be obtained by chemical analysis of ice layers in vast glaciers formed over long time periods. These layers preserve sulfur compounds that steeled onto the ice surface from the atmosphere after explosive volcanic

[1] Use of the radiocarbon dating technique has established the dates of three of Mount Tambora's eruptions prior to the 1815 eruption. The estimated dates are 3910 BC ± 200 years, 3050 BC and 740 AD ± 150 years.

eruptions. These data show that a great explosive eruption occurred 536 AD. Historical chronicles of the late antique and Nordic myths in the Nordic saga confirm the formation of slightly transparent haze in the atmosphere in that year that remained for 1-3 years. Analysis led to the conclusion that it occurred in the tropics (the explosion of Rabaul on the island of New Britain of New Guinea) and that the aerosol cloud produced by the eruption was twice as dense as the aerosol layer resulting from the Tambora eruption (Stothers 1984). It is possible that the climatic change after the eruption of the volcano on the island of Santorin in the eastern part of Mediterranean sea approximately 1500 BC was very significant. It is believed that it led to an abrupt decline of the highly developed Cretan civilization that had flourished up to that time. US National Research Council (1985) (Francis 1983) believes that large eruptions with a volume of aerosol particles ejection greater by one order of magnitude than the ejection of Tambora can produce an aerosol mass of about 10^15 grams. The estimated average temperature decline is around 10 C. It is absolutely certain that an abrupt cooling by tens of degrees would lead to the death of the majority of animals and plants inhabitating the land.

In general, the eruption of Tambora deserves attention because it is the eruption nearest in time that induced a climate change causing noticeable damage to living nature.

1.2 The role of stochastic theory when analyzing grain prices

On a smaller scale price and consumption stability was an important theme in pre-industrial political economy. Historically the quest for price stability was a quest for consumption stability which had welfare-increasing effects. Price shocks threaten the entire fabric of early modern society, by leading to increase in crime and the contribution to the spread of epidemics due to waves of migrants. The consumption stability has indeed attracted renewed interest from the modern analysis of poverty and famines in underdeveloped countries. Local supply shocks were substantial and had a great impact on local prices when markets were poorly integrated and because grain storage was not great enough to smooth variations in harvest. High transport costs prevented an adequate level of trade and the risky nature of carry-over speculations made inventories of grain storage too low. However, marked integration steadily improved over the centuries. The free trade which the eighteenth century political economics favored was a good solution provided transport cost fell. A good harvest solution in one region was usually matched by a shortage somewhere else. The actual output varied from year to year around the target output because of local uncontrollable climatic events or other natural accidents (Persson 1999).

To account for rare events stochastic theories are important for modeling. Stochastic theories model systems which develop in time and space in accordance with probabilistic laws[2]. Briefly, the usual situation is to have a set of random variables $\{ X_t \}$ defined for all values of the real number t (assume time), which could be discrete or continuous. The outcome of a random variable is a state value (often a real number). The set of random variables is called a stochastic

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[2]The space is not necessarily the familiar Euclidean space for everyday life. In this article "space" is prices. We distinguish between cases which are discrete and continuous in time or space. See Taylor and Karlin (1998) for a mathematical definition of stochastic processes, which is not replicated here.

process, which is completely determined if the joint distribution of the set of random variables $\{X_t\}$ is known. A realization of the stochastic process is an assignment to each t in the set $\{X_t\}$ a value of X_t.

Essential in stochastic theories is how randomness is accounted for (Moxnes and Hausken 2010a). A noise could be Gaussian or more general, and it could be uncorrelated (white noise) or correlated (colored noise). A simple realization could be to let the state value at each time to be a random noise or alternatively a deterministic function of time added a random noise as in Markov (1906) process. For Markov (1906) processes, the state value of $X_{t+\Delta t}$ at time t+Δt is given by the state value at time t, plus a state value of a "random variable" at time t. A Markov process may be deterministic, i.e. all values of the process at time t'> t are determined when the value is given at time t. Or a process may be nondeterministic, i.e. a knowledge of the process at time t is only probabilistically useful in specifying the process at time t' > t. The influence (correlation) may decrease rapidly as the time point moves into the future. The increment in the Einstein- Smoluchowski theory of Brownian motion during a time interval Δt is proportional with the so called drift, plus a non deterministic noise as a time uncorrelated Gaussian random term (called Gaussian white noise), with mean zero and variance proportional to.[3] Notice that in this additive case, although a noise is uncorrelated, the random variables $\{X_t\}$ are not uncorrelated.

A system with time uncorrelated noise is usually just a coarse-grained version of a more fundamental microscopic model with correlation. It is therefore of interest to study models with correlated noise. Bag et al. (2001) introduced correlation by increasing the number of differential equations and applying uncorrelated noise throughout. In the Stratonovich (1966) method noise is incorporated in a deterministic equation by adding a deterministic noise term that can be integrated in the normal Riemann sense[4]. The Stratonovich integral occurs as the limit of time correlated noise when the correlation time of the noise term approaches zero. However, depending on the problem in consideration, the Stratonovich (deterministic noise) or the Ito model (stochastic noise) could be appropriate approximations. Indeed, for additive noise the Ito model gives the same answer as the Stratonovich model if the Stratonovich model uses a Gaussian distribution for the noise. But for multiplicative noise the results are different. (For a recent treatment of different interpretations of stochastic differential equations see for example Lau and Lubensky (2007).) Roughly, the reason for this difference is that the Stratonovich integral, which is a Riemann integral, applies on functions with bounded variations. The Ito integral applies on functions without bounded variations (i.e. white noise).

[3] This mathematical model was first treated rigorously by Ito (1951) See Arnold (1973) and Oksendal (1985). The work by Bachelier (1900), Einstein (1905, 1926), Smoluchowski (1906,1923), Wiener (1923), Ornstein-Uhlenbeck (1930).

[4] Allowing randomness in the initial values e.g. for prices and for a deterministic noise commonly implies more realistic models of physical situations than e.g. the Liouville process where a realization of the stochastic process is constructed deterministically allowing randomness only in the initial conditions of the prices.

1.3 This paper's contribution

There are numerous methods to analyze stochastic processes as the data of time series[5]. The characteristic properties of a time series is that the data are not generated independently, their dispersion varies with time, and they are often governed by a trend that could have cyclic components. In this article we first apply a statistical hypotheses test on historical time series data of the prices of grain in England during the periods 1750-1789 and 1821-1850. The period 1790-1814 during the Napoleon war is considered as a separate case. We study the Mount Tambora volcano eruption period 1815-1820 to examine whether the grain prices were significantly higher than during the two other periods examined. Next we apply a simple regression analysis. Finally we compare realizations of different stochastic processes with historical time series data. We construct an additive model of time series with identical distribution. The trend is piecewise linear with no cyclic component.

The purpose of the paper is to test stochastically the available time series data of the England 1750-1850 grain prices. We focus explicitly on how a period of crisis during this period, the 1815-1820 *Tamboro* volcano eruption period, affected the grain prices. We seek to learn from this crisis period to develop policy implications for how to handle future crisis.

1.4 This paper's organization

Section 2 considers hypothesis testing, regression, and an additive zeroth order process with uncorrelated and correlated Gaussian noise. Section 3 considers a first order process with additive Gaussian noise with and without out correlation. Section 4 compares the processes in section 2 and section 3. Section 5 suggests a methodology for policy recommendations Section 6 concludes.

2 A zero'th order process with uncorrelated and correlated additive Gaussian noise.

Statistical Hypothesis tests: Figure 2.1 shows the prices of grain (the average of Wheat, Barley and Rye) (Clark 2007). The red points are for the time period 1790-1814 (Napoleon war, conceived as a special time period). As the simplest analysis we examine the probability that the historical prices for the time period 1815-1820 come from the same distribution (same expectation and variance) as the distribution for the time period 1821-1850. The probability is 10^(-6) assuming a student t distribution.

Statistical regression: Hypothesize that the price of grain at each time is Gaussian distributed and that the stochastic variables X_i (prices) at each time t_i are independent. It appears from the data that a linear function of time could be a feasible approximation for the expectation, i.e. $E(X_i) = a + bt_i$, where t_i is the time, and b most likely is positive during the time period 1750-1789 and negative during the time period 1821-1850. We assume the joint distribution $\rho_{X_1,X_2..X_n}(x_1, x_2,..x_n)$

[5] Statistical procedures that suppose independent and identically distributed data are however in general excluded from analysis of time series.

$$\rho_{X_1,X_2..X_n}(x_1,x_2,..x_n) \overset{mod}{=} \rho_{X_1}(x_1)\rho_{X_2}(x_2)...\rho_{X_n}(x_n), \rho_{X_i}(x_i) \overset{mod}{=} \frac{1}{\sqrt{2\pi}\sigma_i} Exp\left(-\frac{(x_i-a-bt_i)^2}{2\sigma_i^2}\right), \quad (2.1)$$
$$\sigma_1 \overset{mod}{=} \sigma_2 \overset{mod}{=} ...\sigma_n \overset{def}{=} \sigma$$

where "mod" means model assumption and " def" means definition. We assume the same variance at each time. We introduce the well-known maximum likelihood function $l(a,b,\sigma) = \left(1/\left(\sqrt{2\pi}\sigma\right) Exp\left(-(\tilde{x}_i - a - bt_i)^2/(2\sigma^2)\right)\right)^n$, where $\tilde{x}_i$ is the actual historical price at the time t_i and σ^2 is the variance. Requiring that this function is a maximum as a function of all the three variables, it follows that

$$\frac{\partial}{\partial a} Ln\left(l(a,b,\sigma)\right) = 0, \frac{\partial}{\partial b} Ln\left(l(a,b,\sigma)\right) = 0, \frac{\partial}{\partial \sigma} Ln\left(l(a,b,\sigma)\right) = 0$$
$$\Rightarrow na + n\overline{t}b = n\overline{\tilde{x}}, n\overline{t}a + \sum_{i=1}^{n} t_i^2 = \sum_{i=1}^{n} t_i\tilde{x}_i, -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n}(\tilde{x}_i - a - bt_i)}{\sigma^2} = 0 \quad (2.2)$$
$$\overline{t} \overset{def}{=} \frac{1}{n}\sum_{i=1}^{n} t_i, \overline{\tilde{x}} \overset{def}{=} \frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i$$

The solution becomes

$$b = \frac{\sum_{i=1}^{n}(t_i - \overline{t})\tilde{x}_i}{\sum_{i=1}^{n}(t_i - \overline{t})^2}, a = \overline{\tilde{x}} - b\overline{t}, \sigma^2 = \sum_{i=1}^{n}\frac{(\tilde{x}_i - a - bt_i)^2}{n} \quad (2.3)$$

Inserting the data implies $a = a_1$=-60.14, $b = b_1$= 0.036/year during $1750 \le t \le 1789$, and a=a_2=33.82, b=b_2= -0.016/year during $1821 \le t \le 1850$. The standard deviation becomes $\sigma = \sigma_1$ = 0.57 during 1750-1789, and $\sigma = \sigma_2$ = 0.71 during 1821-1850. Note that a and b would be the same if we had minimized the square distance to the prices by two linear regression curves in time (one straight line for each period).

Time series analysis: We set that the expectation E() of the noise is zero, and for more mathematical generality we assume that there is exponential correlation $E\left(\xi(t)\xi(t')\right) \overset{mod}{=} \xi_0^2/(2\tau)Exp\left(-|t-t'|/\tau\right)$, where τ is a correlation time parameter. In the limit when τ approaches zero we achieve uncorrelated noise, to read $Lim_{\tau\to 0}\xi_0^2 Exp\left(-|t-t'|/\tau\right)/(2\tau) = \xi_0^2\delta(t-t')$, where $\delta(\ )$ is the Dirac delta function. We have for period j=1 (1750-1789) and period j=2 (1821-1850)

$$X(t) \overset{mod}{=} \alpha_j + \beta_j t + \underbrace{\xi(t)}_{noise} \Rightarrow E\left[X(t)\right] = \alpha_j + \beta_j t, \quad j = 1, 2, (a)$$
$$E\left[\xi(t)\right] \overset{mod}{=} 0, (b), E\left[\xi(t)\xi(t')\right] \overset{mod}{=} \xi_0^2 /(2\tau) Exp\left(-\left|t - t'\right| / \tau\right), (c) \tag{2.4}$$

We next apply a Stratonovich (1966) model. This gives for each period

$$E\left[X(t)\right] = \alpha_j + \beta_j t$$
$$Cov\left[X(t), X(t')\right] = E\left[\left(X(t) - E\left[X(t)\right]\right)\left(X(t') - E\left[X(t')\right]\right)\right] = E\left[\xi(t)\xi(t')\right] = \frac{\xi_0^2}{2\tau} Exp\left(-\left|t - t'\right| / \tau\right)$$
$$Var\left[X(t)\right] = Cov\left[X(t), X(t)\right] = \xi_0^2 /(2\tau)$$
$$Corr\left(X(t), X(t')\right) = \frac{Cov\left[X(t), X(t')\right]}{Var\left[X(t)\right]^{1/2} Var\left[X(t')\right]^{1/2}} = Exp\left(-\left|t - t'\right| / \tau\right) \tag{2.5}$$

Equation (2.5) shows that the expectation varies linearly with time in each period, while the variance is constant through time. The correlation approaches zero if the correlation time τ approaches zero for every finite time difference $\left|t - t'\right|$.

In principle the expectation and variance of the time series must be found by studying many time series (realizations). However, we have only one historical time series for use and in addition only a finite number of time points. The objective is thus to construct an estimator for α_i, β_i, ξ_0^2 and τ that best fits the available data. An estimate for α_i and β_i is simply found by a least square fit of a linear function to the historical data for each period. Indeed, we have already found those since we have that $\hat{\beta} = b, \hat{\alpha} = a$, where the superscript "^" expresses estimator.

The situation for the variance is more challenging. If historical data values are pairwise uncorrelated, estimated variance is commonly set to

$$Var\left[X(t)\right] = \xi_0^2 /(2\tau) \approx \hat{\sigma}^2 = \sum_{i=1}^{n} \left(X(t_i) - \hat{\alpha} - \hat{\beta} t_i\right)^2 / n \tag{2.6}$$

which is the variance in (2.3). However, some correlation from year to year can possibly be observed in figure 2.1. It could be conceived that if prices are high they tend to stay high and if the prices are low they tend to stay low. The correlation in prices could be explained by storage of grain. It is known that in the old society large amounts of stored grain per unit used grain per year was common, as a precaution against periods of famine, drought, etc. The procedure here is to construct an estimated value for the variance by using (2.6). An alternative would be a least square fit of a constant function to the historical data.

Figure 2.1 shows the expectation together with the confidence levels $1\hat{\sigma}$ and $2\hat{\sigma}$ which correspond to a probability of 0.67 and 0.95 for a Gaussian distribution. We see that the linear fitted curve from 1750-1789, when extrapolated to 1820, is in very good agreement with the linear curve fitted to the points 1821-1850. This is indeed a check on consistency of our assumptions. The standard deviation is estimated by (2.6) both for 1750-1789 and 1821-1850 and become 0.57 and 0.71 respectively. Thus the confidence bands are somewhat wider after the Napoleon war compared to before the Napoleon war (0.57 vs 0.71). (The abnormal Napoleon war period 1790-1814 is not analyzed.) However, the levels are roughly of the same magnitude. During the 1815-1820 volcano eruption period the price of grain is much larger (above the 3 sigma level) than expected from the 1750-1789 or from the 1821-1850 periods[6]. Thus the prices during the volcano eruption period 1815-1820 are significantly larger than the prices during the periods 1750-1789 and 1821-1850. The green curve is a realization of the stochastic process in (2.4). We use no correlation and a Gaussian distribution for the noise with zero expectation and standard deviation equal to the estimated standard deviation of the historical data from 1750-1789 (0.57). The expected curve used in the realization is the blue piecewise straight line with the break in 1815. The green curve 1790-1814 is what we roughly find the grain prices should have been without the Napoleon war. We observe that the green curve nicely fits into the historical data 1821-1850. However, 1815-1820 is abnormal and is not captured by the realization.

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[6] We should mention that the so called Grain Laws were implemented through the period 1814 to 1846. These were “taxes” to ensure minimum price of grain. In 1813, a House of Commons Committee recommended excluding foreign-grown corn until domestically grown corn reached £4 (2010: £202.25) per quarter (1 quarter = 480 lb / 218.8 kg). Thomas Malthus believed this to be a fair price and that it would be dangerous for Britain to rely on imported corn as lower prices would reduce labourers' wages, and manufacturers would lose out due to the fall in purchasing power of landlords and farmers. David Ricardo believed in free trade so Britain could use its capital and population to her comparative advantage. With the advent of peace in 1814, corn prices dropped, and the government passed the 1815 Corn Law.

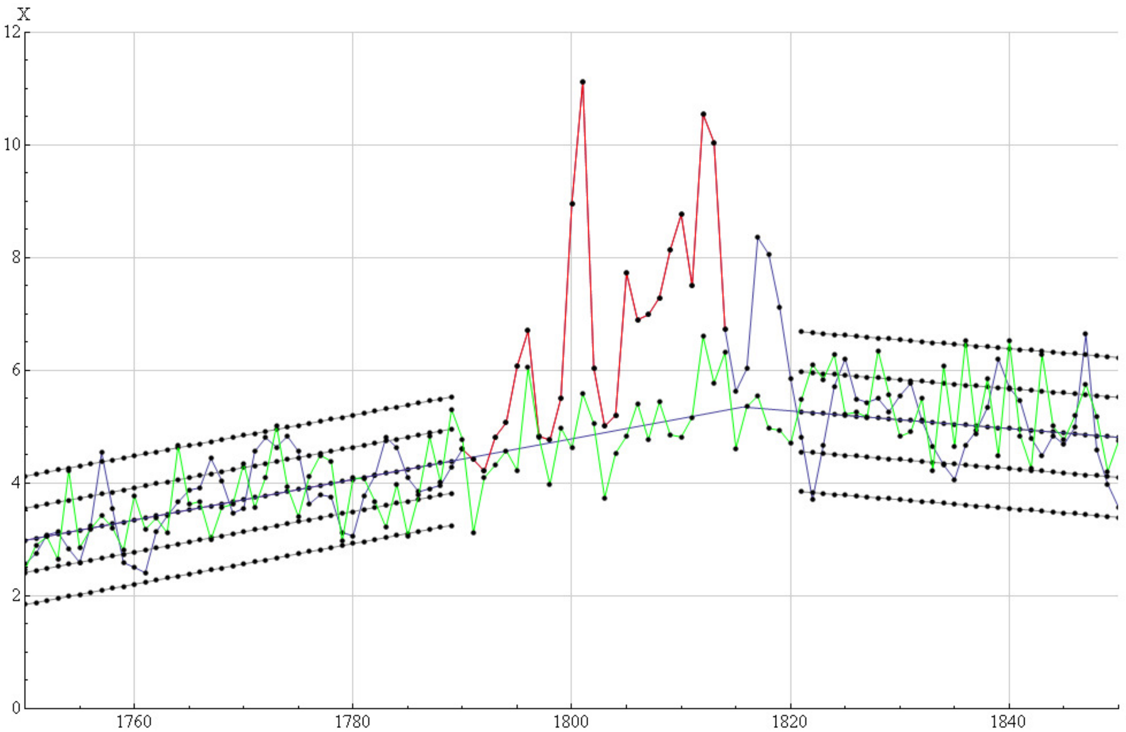

Figure 2.1: The price (in 12 pence per bushel) of grain X(t) in England as a function of time t during 1750-1850. Red and Blue scattered curves are historical data prices as average of Barley, Wheat and Rye. Red curve is prices during the Napoleon war.
Straight lines are expectation and confidence level of 0.67 and 0.95 assuming a Gaussian distribution of noise.
Green curve is a numerical realization of the process in (2.4) using the piecewise linear curve in the middle as expectation.

$$\hat{\alpha} = \hat{\alpha}_1 = -60.14, \hat{\beta} = \hat{\beta}_1 = 0.036 / \, year, \hat{\sigma} = \hat{\sigma}_1 = 0.57, (1750-1814)$$
$$\hat{\alpha} = \hat{\alpha}_2 = 33.82, \hat{\beta} = \hat{\beta}_2 = -0.016 / \, year, \hat{\sigma} = \hat{\sigma}_1 = 0.57, (1815-1850).$$

It is of interest to find the covariance of prices (or correlation) between subsequent years. An estimator or the covariance is thus needed. As a first guess we use the well-known empirical covariance or correlation, to read

$1750-1789:$

$$Cov\left[X(t), X(t')\right] \approx \sum_{i=1}^{40-j} \frac{\left(X(t_i) - \hat{\alpha} - \hat{\beta} t_i\right)\left(X(t_{i+k}) - \hat{\alpha} - \hat{\beta} t_{i+k}\right)}{(40-j)}, t = t_i, t' = t_i + j, t - t' = k = 1, 2..30\left(days\right)$$
$$Corr\left[X(t), X(t')\right] = Cov\left[X(t), X(t')\right] / \hat{\sigma}^2, \tag{2.7}$$

To further check on variance and correlation calculations we set up a simulation according to (2.4) as

$$X(t_i) = \hat{\alpha} + \hat{\beta} t_i + Random[NormalDistribution[0, \hat{\sigma}]], X(t_0) = 2.98$$
$$\hat{\alpha} = \hat{\alpha}_1 = -60.14, \hat{\beta} = \hat{\beta}_1 = 0.036 / year, \hat{\sigma} = \hat{\sigma}_1 = 0.57, (1750 - 1814) \quad (2.8)$$

Next we use (2.6) and (2.7) with the simulated realizations as input. By comparing the output with the input we make a quality check on the estimator for variance and covariance. With the numerical realizations as input in (2.6) for the period 1750-1789 we find the estimated standard deviation of around 0.6 which in good agreement with the input value of $\hat{\sigma} = 0.57$.

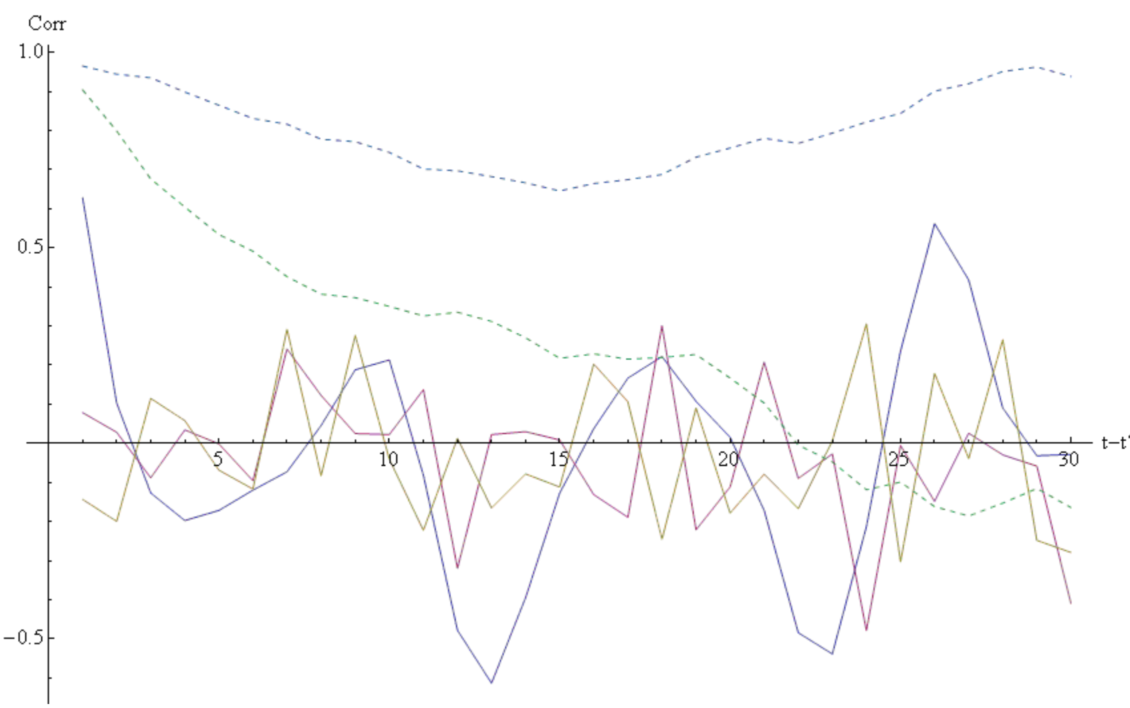

Figure 2.2: The estimated correlation as a function of the time difference in the period 1750-1789.
Blue curve is based on historical data. Yellow and Pink curves are two realizations of the process in (2.8).
The two dotted curves are according to two realizations of the stochastic process in section 3, see (3.9).

In figure 2.2 we see two correlation calculation functions based on two realizations of the process in (2.4). The correlations are scattered around zero for the time difference of $t - t' = 1$-

30. This is exactly what they should be since the correlation time was set to zero in the realizations. The historical data show a somewhat smoother curve oscillating around zero. Some correlation seems to be operative. It could also be that some cyclic behavior is present in the historical price data. We have performed a Fourier Sine discrete transform of the historical data and three peaks appear in the frequency domain. However, roughly it appears that the correlation estimator in (2.7) is applicable both for the realization and possible also for the historical data.

We also apply the prices during the Napoleon war in the study by fitting a linear regression curve over the complete period 1750 to 1850. We obviously find a much broader confidence band as seen in figure 2.3. The standard deviation is 1.54. Now it appears that the historical data points of the period 1815-1820 are inside the 0.95 confidence limit.

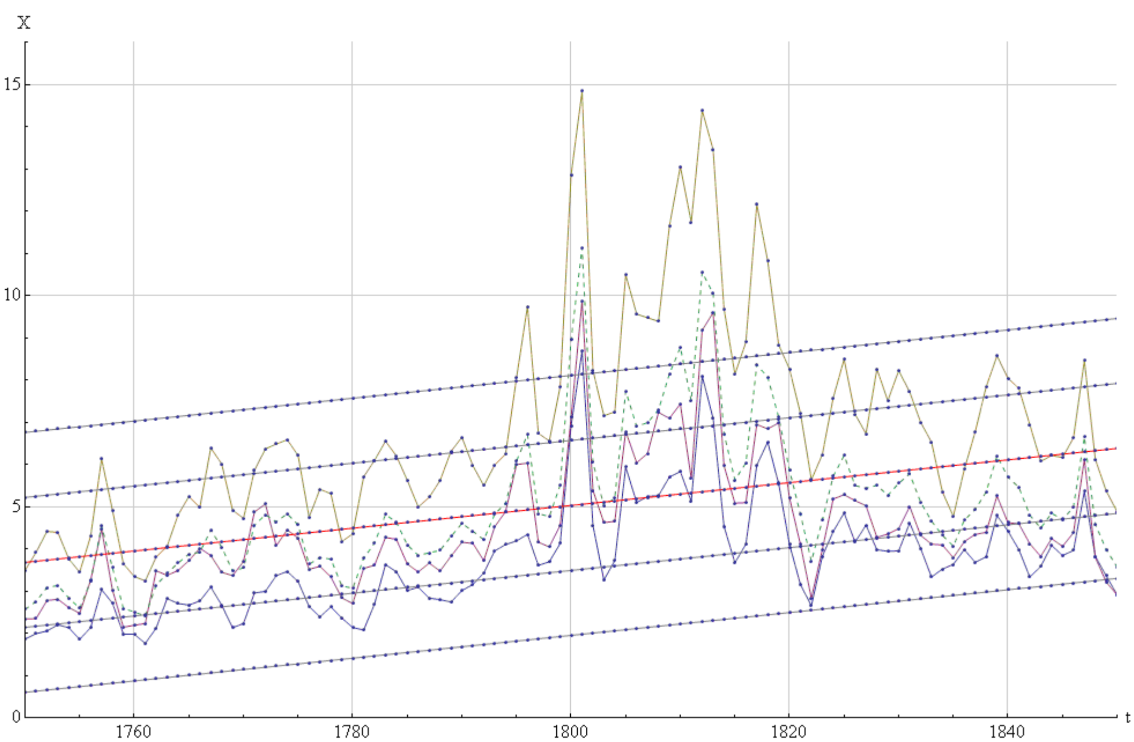

Figure 2.3: The price (in 12 pence per bushel) X(t) of grain as a function of time t.
Green dotted curve is the average of Wheat Barley and Rye from historical data.
Yellow curve is Wheat. Blue curve is Barley. Pink curve is Rye.
Red straight curve is expectation found by linear least square fitting to the average price of Wheat, Rye and Barley over the period 1750-1850. Confidence bands of 0.67 and 0.95 are also shown.

3 A first order process with uncorrelated and correlated additive Gaussian noise

Assume that dX_t is the change in the price during a small time interval from t to t+h . In the study of nonlinear recurrence relations Mackey and Glass (1988) showed that it is possible to construct an infinite number of deterministic relations which are chaotic, but which describe a given density distribution. Thus a given density distribution has no unique recurrence relation. It appears that the broad class of Markovian theories incorporating the Gaussian white noise input provides a satisfactory approximation for a large variety of phenomena. We consider more generally a stochastic differential equation with additive noise of the Ito form

$$\begin{aligned} &dX_t \overset{def}{=} X_{t+h} - X_t \overset{mod}{=} f(X_t)h + d^*\varsigma(t,h,X_t), \\ &d^*\varsigma(t,h,X_t) \overset{mod}{=} Random\left[Distribution\left[0, d\sigma_2(t,h,X_t)^{1/2}\right]\right],(a) \\ &E\left(d^*\varsigma(t,h,X_t)\right) \overset{mod}{=} 0,(b), E\left(H(X_t,t)d^*\varsigma(t,h,X_t)\right) \overset{mod}{=} 0,(c) \end{aligned} \tag{3.1}$$

$d^*\varsigma(t,h,X_t)$ is not a differential in the Riemann sense and is therefore denoted with the "*". $\sigma_2(t,h,X_t)$ is a function and $d\sigma_2(t,h,X_t)$ is the differential change of $\sigma_2(t,h,X_t)$ with respect to h during the time step h. We model $d^*\varsigma(t,h,X_t)$ as a stochastic variable where *Distribution* is any distribution of expectation zero and variance $E\left(d^*\varsigma^2\right) = d\sigma_2$. E() means expectation. $H(x,t)$ is any arbitrary function. The expectation of $d^*\varsigma(t,h,X_t)$ in (3.1b) is zero. Equation (3.1c) is valid when using Ito calculus (no bounded variation), but is not valid when using Stratonovich calculus. When $d^*\varsigma(t,h,X_t)$ =0, we define, as determined by (3.1), what we call a Liouville recurrence relation, where only the initial values are stochastic. We assume that a) the variance is $E\left(d^*\varsigma^2\right) = d\sigma_2 \overset{mod}{=} g_2(t,X_t)dt = g_2(t,X_t)h$ for small time steps, where $g_i(t,X_t)$, i=2,3,4,…, is some well behaved function to be specified exogenously, and b) that higher order moments also have the same powers of h, akin to multifractal phenomena (Stanley and Meakin 1988)[7]. Thus for n>1, $d\sigma_n = E\left(d^*\varsigma^n\right) \overset{mod}{=} g_n(t,X_t)h, \Rightarrow \dot{\sigma}_n = g_n(t,X(t))$. By applying Ito calculus with $f(X_t) = \beta$ it follows (Moxnes and Hausken 2010ab)

$$\dot{\rho}(t,x) = -D\left(\rho(t,x)\beta\right) + (1/2!)D^2\left(\rho(t,x)\dot{\sigma}_2(t,x)\right) + (1/3!)D^3\left(\rho(t,x)\dot{\sigma}_3(t,x)\right) + ... \tag{3.2}$$

In fact, we apply that our uncorrelated random term is Gaussian, i.e., all odd moments are zero, and even moments of higher order than two are of higher order than h (see appendix A). This gives $\dot{\sigma}_3(t,x) = \dot{\sigma}_4(t,x) = \dot{\sigma}_5(t,x) = ... = 0$, implying the partial differential equation called the

[7] Traditionally only the Gaussian distribution is used

forward Fokker-Planck equation or forward Kolmogorov equation (Risken 1989). We can from (3.2) easily calculate the derivative of the expectation and variance of X_t which for a time dependent (and uncorrelated) random term $\dot{\sigma}_2 = \dot{\sigma}_2(t)$ is

$$\partial/\partial t\left(E\left(X_t\right)\right) = \beta, \partial/\partial t \ Var\left(X_t\right) = \dot{\sigma}_2 \tag{3.3}$$

We now formulate a continuous equation in time that corresponds to (3.2), to read $\dot{X}(t) = f(X(t)) + \xi(t) = f(X(t)) + \dot{\varsigma}(t), \quad \dot{\varsigma}(t) = d\varsigma/dt \overset{def}{=} \xi(t)$, where $\xi(t)$ an arbitrary noise function is. We let $\xi(t)$ be deterministic (the randomness is then only in the initial values of $\xi(t)$; see example in appendix C). The deterministic approach generally gives that $\xi(t)$ can be integrated in the traditional Riemann sense. The Stratonovich integral occurs as mentioned in the limit of colored noise if the correlation time of the noise approaches zero. A quite common and different integral is the Ito integral for the uncorrelated situation and Gaussian noise. To match the noise in (3.1) we set that expectation is zero and that the noise is uncorrelated, i.e. $E\left(\xi(t)\xi(t')\right) \overset{mod}{=} \dot{\sigma}_2(t)\delta\left(t-t'\right)$, $\delta()$ is the Dirac delta function that accounts for the lack of correlation. In addition we will apply that the noise is Gaussian. However, we will be methodically more general, and allow for correlation in the noise, to read in the Stratonovich sense

$$\begin{aligned} &\dot{X}(t) = \beta + \xi(t) \Rightarrow X(t) = X\left(t_0\right) + \int_{t_0}^{t} \left(\beta + \xi(t')\right) dt', (a) \\ &E\left[X(t)\right] = E\left[X(t_0)\right] + \beta(t - t_0), (b) \\ &E\left[\xi(t)\right] \overset{mod}{=} 0, (c), 0, (c) \end{aligned} \tag{3.4}$$

This gives the covariance

$$\begin{aligned} &Cov\left[X(t), X(t')\right] = E\left[\left(X(t) - E\left[X(t)\right]\right)\left(X(t') - E\left[X(t')\right]\right)\right] \\ &= E\left[\left(X\left(t_0\right) - E\left[X(t_0)\right] + \int_{t_0}^{t} \xi(u)du\right)\left(X\left(t_0\right) - E\left[X(t_0)\right] + \int_{t_0}^{t'} \xi(v)dv\right)\right] \\ &= Var\left[X(t_0)\right] + \int_{t_0}^{t} \int_{t_0}^{t'} E\left(\xi(u)\xi(v)\right) dudv) \end{aligned} \tag{3.5}$$

We have used that the covariance of the initial value $X(t_0)$ and the noise is zero, to read $E\left(X(t_0)\xi(u)\right) = E\left(X(t_0)\right)E\left(\xi(u)\right) = 0$. The integral in (3.5) can be calculated explicitly for a time dependent (non stationary) uncorrelated noise where $E\left(\xi(t)\xi(t')\right) = \dot{\sigma}_2(t)\delta(t-t'), \dot{\sigma}_2(t) = \xi_0^2\left(1 - Exp\left(-(t-t_0)/\tau\right)\right)$, and also for correlated noise as $E\left(\xi(t)\xi(t')\right) = \xi_0^2/(2\tau)Exp\left(-\left|t-t'\right|/\tau\right)$. We find when inserting into (3.5) that (Moxnes and Hausken 2010ab)

$$Cov\left[X(t),X(t')\right]=\begin{cases}Var\left[X(t_0)\right]+\xi_0^2\left[t'-t_0+\tau\left(Exp\left(-(t'-t_0)/\tau\right)-1\right)\right],: uncorrelated\ noise\\ Var\left[X(t_0)\right]\\ +\xi_0^2\left(t'-t_0+\frac{\tau}{2}\left(Exp\left(-(t'-t_0)/\tau\right)-1-Exp\left((t'-t)/\tau\right)+Exp\left(-(t-t_0)/\tau\right)\right)\right),\\ correlated\ noise\end{cases}$$

$$Var\left[X(t)\right]=Var\left[X(t_0)\right]+\xi_0^2\left(t-t_0+\tau\left(Exp\left(-(t'-t_0)/\tau\right)-1\right)\right),: uncorrelated\ and\ correlated\ noise$$

$$Corr\left[X(t),X(t')\right]\overset{def}{=}\frac{Cov\left[X(t),X(t')\right]}{Var\left[X(t)\right]^{1/2}Var\left[X(t')\right]^{1/2}}$$

(3.6).

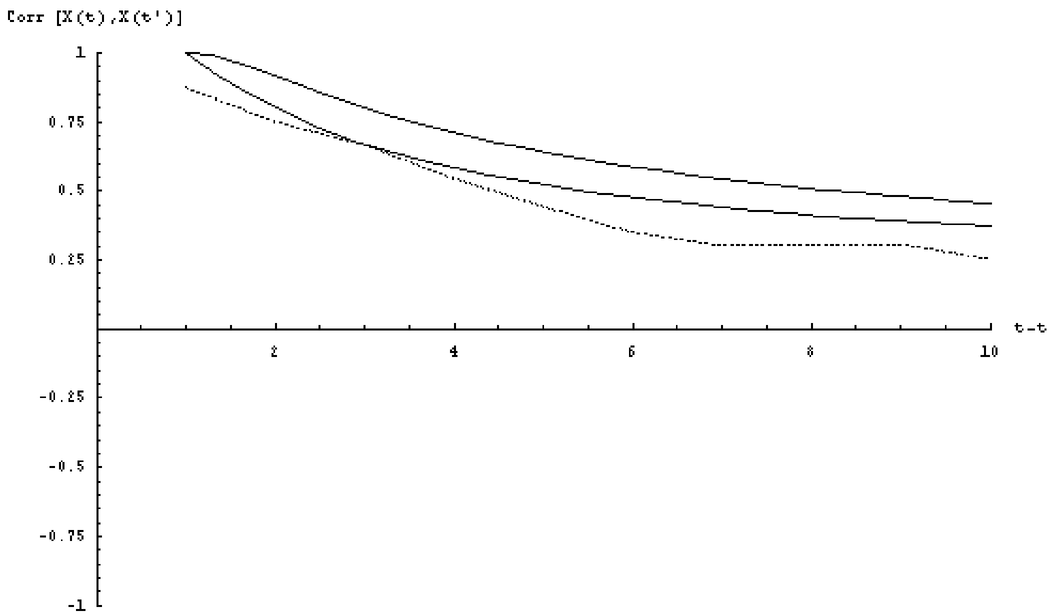

Figure 3.1: The correlations as function of the time difference t-t'. $Var\left[X(t_0)\right]=\hat{\sigma}_1^{\,2}$ *= 0.57^2, t'=0. The dotted line is constructed from a realization according to (2.7). The drawn lines are according to (3.6) for correlated noise (upper) and uncorrelated noise (lower) processes.* $\xi_0^2=Var\left[X(t_0)\right]=\hat{\sigma}_1^{\,2},\tau=1$.

Notice in (3.6), in contrast to the model in section 2, that the first order process is correlated despite the noise being uncorrelated. Figure 3.1 shows the two correlation functions when $\xi_0^2=Var\left[X(t_0)\right]=\hat{\sigma}_1^{\,2},\tau=1$.

That the correlated and uncorrelated noise in (3.6) shows the same variance was not by chance. Indeed, to mimic the correlated noise by uncorrelated noise we must have $\int_{t_0}^{t}\int_{t_0}^{t} E\left(\xi(u)\xi(v)\right)dudv = \int_{t_0}^{t}\int_{t_0}^{t} \dot{\sigma}_2(u)\delta(u-v)dudv$. A sufficient condition is

$$\dot{\sigma}_2(t) = 2\int_{t_0}^{t} E\left(\xi(u)\xi(t)\right)du \qquad (3.7).$$

We easily find for our case

$$2\int_{t_0}^{t} E\left(\xi(t)\xi(u)\right)du = \frac{\xi_0^2}{\tau}\int_0^t Exp\left(-\left|u-t\right|/\tau\right)du = \xi_0^2\left(1-Exp\left(-(t-t_0)/\tau\right)\right) = \dot{\sigma}_2 \qquad (3.8)$$

Gaussian processes imply that higher order moments follow from the second order moment. Thus if the variances are equal, the two Gaussian processes are equivalent, despite one being correlated and the other being uncorrelated.

4 Comparing the processes in section 2 and section 3

This section compares the zeroth order process in section 2 with uncorrelated noise, with the first order process in section 3 with uncorrelated noise (that is, $\tau \to 0$). For this purpose we consider what we perceive to be the most plausible special case of (3.1) where

$$\begin{aligned}
&X\left(t+h\right) = X\left(t\right) + \hat{\beta}h + Random\left[NormalDistribution\left[0,\hat{\sigma}h^{1/2}\right]\right], X\left(t_0\right) = E\left[X\left(t_0\right)\right] = 2.98,\\
&h = 0.01 year,\\
&\hat{\alpha} = \hat{\alpha}_1 = -60.14, \hat{\beta} = \hat{\beta}_1 = 0.036/\,year, \hat{\sigma} = \hat{\sigma}_1 = 0.57, (1750-1814)\\
&\hat{\alpha} = \hat{\alpha}_2 = 33.82, \hat{\beta} = \hat{\beta}_2 = -0.016/\,year, \hat{\sigma} = \hat{\sigma}_1 = 0.57, (1815-1850)
\end{aligned} \qquad (4.1)$$

Equation (4.1) states that X(t+h) depends on X(t) plus a term $\hat{\beta}h$ proportional to the time increment h, plus a random term drawn from a Normal distribution with expectation zero and variance $\hat{\sigma}h^{1/2}$. We perceive this process as a nice benchmark and use it to check whether the zeroth and first order processes fit the data.

Figure 4.1 shows two different realizations of the process in section 2, and two different realizations of the first order process in section 3.

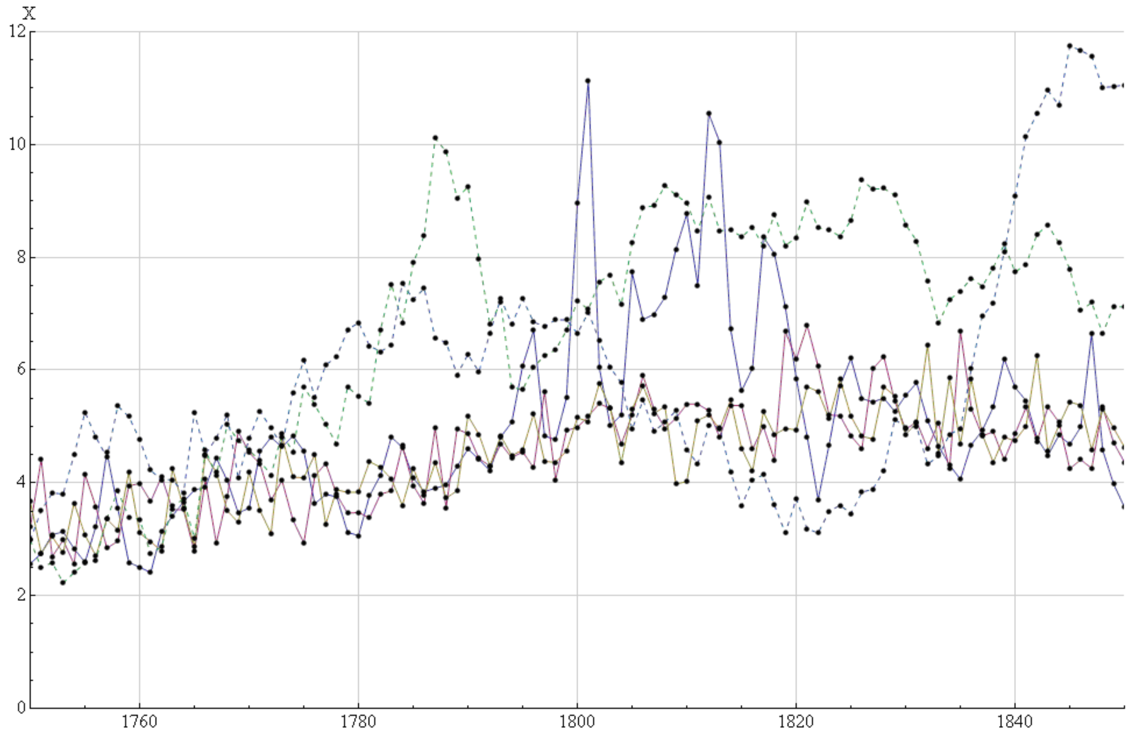

Figure 4.1: Different realizations of price X(t) of grain as a function of time t.
Blue curve is historical data.
Yellow and Pink curve are two realizations of the process in section 2.
Dotted curves are two realizations of the first order process in section 3.

Since the variance is increasing monotonically with time with our first order process, most realizations of the first order process tend to deviate from the linear trend with increasing time. However, some of the realizations will not do that (by luck) and give good agreement with the historical data. For 1815-1820 very few simulations are able to capture the very large fluctuations in the data. However, it appears that the first order process can for some special cases achieve prices above the 1815-1820 periods. However, to reach these prices a prolonged deviation from the average curve over time is needed which is in conflict with the data. It clearly appear that the zero order process fits the data much better than a first order process.

5 A methodology for policy recommendations

This paper has shown that the prices during the volcano eruption period 1815-1820 are significantly (outside the confidence 3 sigma level) larger than the prices during the periods 1750-1789 and 1821-1850. Let us use the result for this crisis to quantify various relationships to develop policy recommendations for future crises.

Observe from Figure 2.1 that the average grain price (in 12 pence per bushel) during the five-year volcano eruption time period 1815-1820 is 6.84±1.18. The average grain price during the years 1750-1789 increases linearly from 3±0.57 to 4.2±0.57. The average grain price during the years 1821-1850 decreases linearly from 5.1±0.71 to 4.8±0.71. The range of these values is from 3 to 5.1. The crisis price increase compared with the low value outside the crisis is determined from 6.84/3 = 2.28 which gives 128% price increase. The crisis price increase compared with the high value outside the crisis is determined from 6.84/5.1 = 1.34 which gives 34% price increase. A price increase 34%-128% for a basic commodity such as grain can be expected to disrupt any society. When quantity decrease price increase. The functional relationship depends on the kind of commodity and consumer preferences. All societies need to prepare itself for periods of crisis. This means developing methodologies of thinking, estimating realistic crisis scenarios, and estimating storage needed to prepare for crises. Let us consider one simple example. The example is easily altered by considering alternative price-quantity relationship and preferences.

Example: Assume $p=1/q$ where p is price and q is quantity. Increasing p from 1 to either 1.34 or 2.28 means decreasing q from 1 to either 0.75 or 0.44. Assume that the average citizen consumes 138 kg grain per year,[8] has the means to purchase this amount, and lives in a society designed to deliver this amount without a crisis. Let us consider the more extreme scenario. The product 0.44×138 kg = 61 kg means that only 73 kg is available per year during the crisis, and hence 77 kg is missing. For a society of 10 million people[9] to remedy this deficiency and prepare for a five-year crisis, storage of $77\times10^7\times5$ kg = 3.85×10^9 kg is needed. A society may either store this quantity, or assess whether plausible substitutes can be applied during a crisis, or whether the missing quantity can be purchased outside the area of crisis if the crisis is local.

The period from 1750 to 1850 covers major changes in UK grain markets, as the country moved from self-sufficiency to needing continuous imports. Between 1793 and 1815 England was constantly at war with various European powers. That this led to high grain prices illustrates that the country now needed to import significant amounts of grain – otherwise the War would have had only minimal impact on domestic prices.

International trade in grain developed rapidly after the Napoleonic Wars, with England as the primary recipient, aided by England transferring workers from the fields to the factories. Grain imports were vital to England's economic growth.

We find that the fundamental price-generating process reappeared mostly equivalently after the Napoleonic Wars and after the Tambora disruption. Notable is that the food situation in the UK changed profoundly after both these two events, moving from self-sufficiency to an inability to feed itself.

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[8] See the "food storage calculator" at http://providentliving.org/content/display/0,11666,7624-1-4071-1,00.html, accessed February 26, 2012.

[9] The England population increased from 6.5 million in 1750, to 10.5 million in 1821, and to 16.8 million in 1851.

6 Conclusion

We study the prices of grain in England 1750-1850, with special attention to the *Tamboro* volcano eruption time period 1815-1820. Two different types of time series are constructed and compared with the historical data. We find that the 1750-1789 time series, before the 1790-1814 Napoleon war, is much like the time series 1821-1850. However, the time series during the 1815-1820 Tamboro volcano eruption period is clearly very different from these two time series. Prices appear significantly outside the confidence 3 sigma level, where sigma is measured for the 1750-1789 period. The prices during the volcano eruption period 1815-1820 are significantly larger than the prices during the periods 1750-1789 and 1821-1850. A fat tail distribution is necessary to explain these values. However, no signature of a fat tail distribution is seen in other parts of the time series. The high prices during the volcano eruption period can be captured by a 2 sigma band if the historical grain prices in the volcano eruption period (1815-1820) are analyzed together (or mixed together) with the very high prices during the Napoleon war. However, we find no reason for mixing the data during the war time period with the peace time period. Our conclusion is that the historical report about increasing food prices due to the volcano eruption is supported by the statistical analysis of the historical data of grain prices. The time series during the 1790-1814 Napoleon war show very high prices and extreme fluctuations and are not analyzed in detail. Prices peaked in 1817, and not in 1816 which was a harsh years known as “the year without a summer”.

The period from 1750 to 1850 covers major changes in UK grain markets, as the country moved from self-sufficiency to needing continuous imports. Between 1793 and 1815 England was constantly at war with various European powers. That this led to high grain prices illustrates that the country now needed to import significant amounts of grain – otherwise the war should have had less impact on domestic prices. It is conceivable that high corn prices together with high import during the period 1815-1820 is explained by the in general high prices on the word market due to the volcanic eruption. It has also been reported that the prices of corn in the Cincinnati market increased significantly some years after the Tamboro volcano eruption. This also underscores that the prices of corn increased dramatically in the whole western world between 1815 and 1819 (Berry 1943, Stommel and Stommel 1979).

Acknowledgement: We thank Ole Gjølberg for useful comments and for providing some of the historical grain prices for England that were used.

Appendix A:

Following Heinrichs (1993), we write

$$\dot{X}(t)=\xi(t)\ ,E\left(\xi(t)\right)\overset{def}{=}\int \underbrace{\rho_{\xi(t)}}_{\substack{Marginal\\ distribution}}(t,\xi(t))d\xi(t)\overset{mod}{=}0,$$

$$Correlation:E\left(\xi(t)\,\xi(t')\right)\overset{def}{=}\int \xi(t)\xi(t')\underbrace{\rho_{\xi(t),\xi(t')}}_{\substack{Joint\\ distribution}}(t,\xi(t),\xi(t'))d\xi(t)d\xi(t') \qquad (A.1)$$

$$E\left(\xi(t)\,\xi(t')\right)\overset{mod}{=}\frac{\xi_0^2}{2\tau}Exp\left(-\left|t-t'\right|/\tau\right)$$

where E() is the expectation. The correlation time is τ. We observe that $Lim_{\tau\to 0}Exp\left(-\left|t-t'\right|/\tau\right)=\delta\left(t-t'\right)$. In the limit of zero correlation time we achieve the uncorrelated noise since the Fourier transform (called spectrum) of the delta function is a constant function (called flat). Integrating (A.1) gives

$$X(t)=\int_0^t \xi(t')dt' \qquad (A.2)$$

The expectation and variance become

$$E\left[X(t)\right]=E\left(\int_0^t \xi(t')dt'\right)=\int_0^t E\left(\xi(t')\right)dt'=0$$

$$Var\left[X(t)\right]=E\left[X(t)^2\right]=E\left(\int_0^t\int_0^t \xi(t')\xi(t'')dt'dt''\right)=\int_0^t\int_0^t E\left(\xi(t')\xi(t'')\right)dt'dt'' \qquad (A.3)$$

$$=\frac{\xi_0^2}{2\tau}\int_0^t\int_0^t Exp\left(-\left|t'-t''\right|/\tau\right)dt'dt''=\xi_0^2\left(t-\tau\left(1-Exp\left(-t/\tau\right)\right)\right)$$

In general

$$\sigma_{2n}\overset{def}{=}E\left(X(t)^{2n}\right),\sigma_n=\int_0^t\int_0^t\int_0^t\int_0^t \xi(t')\xi(t'')\xi(t''')\xi(t'''')....\underbrace{dt'dt''dt'''dt''''...}_{n\ times} \qquad (A.4)$$

The characteristic function for the density $\rho_X(t,x)$ of X(t) becomes

$$\theta(t,k)\overset{def}{=}\int Exp\left(ikx\right)\rho_X(t,x)dx=\int\left(1+(ikx)+(ikx)^2/2!+(ikx)^3/3!+..\right)\rho_X(t,x)dx$$

$$=1+\frac{i^2}{2!}k^2\sigma_2+\frac{i^3}{3!}k^3\sigma_3+\frac{i^4}{4!}k^4\sigma_4+... \qquad (A.5)$$

We assume generally that

$$
\begin{aligned}
&\frac{1}{2\pi}\int Exp\left(-ikx\right)\theta(t,k)dk \\
&=\frac{1}{2\pi}\int Exp\left(-ikx\right)\left(1+\frac{i^2}{2!}k^2\sigma_2+\frac{i^3}{3!}k^3\sigma_3+\frac{i^4}{4!}k^4\sigma_4+...\right)dk \\
&\overset{mod}{=}\frac{1}{2\pi}\int Exp\left(-ikx\right)Exp\left(\frac{i^2}{2!}k^2\sigma_2+\frac{i^3}{3!}k^3\sigma_3+\frac{i^4}{4!}k^4\sigma_4+...\right)dk
\end{aligned}
\tag{A.6}
$$

This gives that

$$
\begin{aligned}
&\rho_X(t,x)=\frac{1}{2\pi}\int Exp\left(-ikx\right)\theta(t,k)dk \\
&\dot{\rho}_X(t,x)=\frac{1}{2\pi}\int Exp\left(-ikx\right)\left(\frac{i^2}{2!}k^2\dot{\sigma}_2+\frac{i^3}{3!}k^3\dot{\sigma}_3+\frac{i^4}{4!}k^4\dot{\sigma}_4+...\right)dk \\
&=\frac{1}{2!}\dot{\sigma}_2D^2\rho_X(t,x)-\frac{1}{3!}\dot{\sigma}_3D^3\rho_X(t,x)+\frac{1}{4!}\dot{\sigma}_3D^4\rho_X(t,x)+...
\end{aligned}
\tag{A.7}
$$

However, by assuming a Gaussian distribution all odd moments become zero. The even moments are given by

$$
\begin{aligned}
&\sigma_{2n}\overset{def}{=}E\left(X(t)^{2n}\right)\overset{Gaussian}{=}\frac{(2n)!}{2^n n!}E\left(X(t)^2\right)^n=\frac{(2n)!}{2^n n!}\sigma_2{}^n, n=1,2,3.. \\
&d\sigma_{2n}\overset{Gaussian}{=}\frac{(2n)!}{2^n n!}(d\sigma_2)^n,
\end{aligned}
\tag{A.8}
$$

Thus we find that

$$
\begin{aligned}
&\theta(t,k)=1+\frac{i^2}{2!}k^2\sigma_2+\frac{i^4}{4!}k^4\sigma_4+\frac{i^6}{6!}k^6\sigma_6+... \\
&=1+\frac{i^2}{2}k^2\sigma_2+\frac{i^4}{2^2 2!}(k^2\sigma_2)^2+\frac{i^6}{2^3 3!}(k^2\sigma_2)^3+...=Exp\left(-k^2\sigma_2/2\right) \\
&\rho_X(t,x)=\frac{1}{2\pi}\int Exp\left(-ikx\right)\theta(t,k)dk=\frac{1}{2\pi}\int Exp\left(-ikx\right)Exp\left(-\frac{k^2\sigma_2}{2!}\right)dk \\
&\dot{\rho}_X(t,x)=\frac{\dot{\sigma}_2}{2!}D^2\rho_X(t,x)
\end{aligned}
\tag{A.9}
$$

The Ito calculus is traditionally applied with Gaussian distribution since this gives continuous realizations. In (A.9) we could have applied a non Gaussian distribution. For this case the

Stratonovich solution will not agree with the Ito solution, even for additive noise. However, the general assumption in (A.6) is very difficult to conceive.

Appendix B: A simple deterministic noise as an example

Say that the noise is given by

$$\ddot{\xi}(t) = -\xi(t) \Rightarrow \xi(t) = a\,Sin(t+\phi) \tag{B.1}$$

where ϕ is a time parameter. Let the density be given by

$$\rho_\phi(\phi) = \begin{cases} \frac{1}{2\pi}\; when\; 0 \le \phi \le 2\pi \\ 0\; otherwise \end{cases} \tag{B.2}$$

This gives when we for simplicity set that a=1

$$\begin{aligned} &P\left(Sin(t+\phi) \le \xi\right) = P\left(\underbrace{ArcSin(\xi)+\pi/2 \le \phi + t}_{\in(0,2\pi)} \le \underbrace{ArcSin(\xi)}_{\in(0,2\pi)}\right) \\ &= 2\int_0^{ArcSin(\xi)-t+n2\pi} \rho_\phi(\phi')\,d\phi', \\ &\rho_\xi(\xi) = \frac{\partial\left(P\left(Sin(t+\phi)\le\xi\right)\right)}{\partial\xi} = \frac{2\rho_\phi\left(ArcSin\left(ArcSin(\xi)-t+n2\pi\right)\right)}{\left(1-\xi^2\right)^{1/2}} \\ &= \frac{1}{\pi\left(1-\xi^2\right)^{1/2}}, |\xi| \le 1, \int_{-\infty}^{\infty} \rho_\xi(\xi')\,d\xi' = 1, E(\xi) = 0, E\left(\xi^2\right) = 1/2 \end{aligned} \tag{B.3}$$

Further we have for t'=t+v, where v is a time parameter,

$$\begin{aligned} &\xi(t') = \xi(t+v) = Sin(t+v+\phi) = Sin(v)Cos(t+\phi) + Cos(v)Sin(t+\phi) \\ &= Sin(v)\left(1-\xi(t)^2\right)^{1/2} + Cos(v)\xi(t) \\ &\rho_{\xi,\xi'}\left(\xi(t'),\xi(t)\right) = \rho_\xi\left(\xi(t)\right)\rho_{\xi'/\xi}\left(\xi(t'),\xi(t)\right), \\ &= \frac{1}{\pi\left(1-\xi(t)^2\right)^{1/2}}\delta\left(\xi(t') - Sin(v)\left(1-\xi(t)^2\right)^{1/2} - Cos(v)\xi(t)\right) \end{aligned} \tag{B.4}$$

This gives that

$$
\begin{aligned}
&Cov\left(\xi(t'),\xi(t)\right)=E\left(\xi(t')\xi(t)\right)=\int \rho_{\xi',\xi}\left(\xi(t'),\xi(t)\right)\xi(t')\xi(t)d\xi(t')d\xi(t)\\
&=\int \frac{\delta\left(\xi(t')-Sin(v)\left(1-\xi(t)^2\right)^{1/2}+Cos(v)\,\xi(t)\right)\xi(t')\xi(t)}{\pi\left(1-\xi(t)^2\right)^{1/2}}d\xi(t')d\xi(t)\\
&=\int \frac{\left(Sin(v)\left(1-\xi(t)^2\right)^{1/2}+Cos(v)\,\xi(t)\right)\xi(t)}{\pi\left(1-\xi(t)^2\right)^{1/2}}d\xi(t)\\
&=\int Sin(v)\,\xi(t)d\xi(t)+\int \frac{Cos(v)\,\xi(t)^2}{\pi\left(1-\xi(t)^2\right)^{1/2}}d\xi(t)=Cos(v)\,E\left(\xi(t)^2\right)=\frac{Cos(t'-t)}{2}
\end{aligned}
\tag{B.5}
$$

Thus the covariance or correlation is cyclic in the time difference.

Appendix C:

Assume as an example that the random term is exponentially correlated, with correlation $E\left(\xi(t)\xi(t')\right)\overset{mod}{=}\xi_0^2/(2\tau)Exp\left(-\left|t-t'\right|/\tau\right)$, where τ is the correlation time. An equation for $\xi(t)$ that in fact will generate this exponentially correlated noise for times t' much lager than τ is

$$
\dot{\xi}(t)=-\frac{\xi(t)}{\tau}+\psi(t) \tag{C.1}
$$

where $\psi(t)$ is white noise with $E\left(\psi(t)\psi(t')\right)=\xi_0^2\delta(t-t')/\tau^2$. We have that

$$
\begin{aligned}
&\dot{\xi}(t)=-\xi(t)/\tau+\psi(t)\\
&\xi(t)=X(0)Exp(-t/\tau)+Exp(-t/\tau)\int_0^t Exp(au)\psi(u)du,t_0=0\\
&E\left(\xi(t)\right)=E\left(\xi(0)\right)Exp(-t/\tau)
\end{aligned}
\tag{C.2}
$$

This gives the covariance as

$$
\begin{aligned}
&Cov\left[\xi(t),\xi(t')\right]=E\left[\left(\xi(t)-E\left(\xi(t)\right)\right)\left(\xi(t')-E\left(\xi(t')\right)\right)\right] \\
&=E\left[\begin{array}{l}\left(\xi(0)Exp\left(-t/\tau\right)-E\left[\xi(0)\right]Exp\left(-t/\tau\right)+Exp\left(-t/\tau\right)\int_0^t Exp\left(u/\tau\right)\psi(u)du\right) \\ \times\left(\xi(0)Exp\left(-t'/\tau\right)-E\left[\xi(0)\right]Exp\left(-t'/\tau\right)+Exp\left(-t'/\tau\right)\int_0^{t'} Exp\left(v/\tau\right)\psi(v)dv\right)\end{array}\right] \\
&=Exp\left(-t/\tau-t'/\tau\right)Cov\left[\xi(0),\xi(0)\right] \\
&+Exp\left(-t/\tau-t'/\tau\right)E\left[\xi(0)\int_0^{t'} Exp\left(v/\tau\right)\psi(v)dv+\xi(0)\int_0^t Exp\left(u/\tau\right)\psi(u)du\right] \\
&+Exp\left(-t/\tau-t'/\tau\right)E\left[\int_0^t\int_0^{t'} Exp\left(u/\tau+v/\tau\right)\psi(u)\psi(v)dudv\right] \\
&=Exp\left(-t/\tau-t'/\tau\right)Var\left[\xi(0)\right]+Exp\left(-t/\tau-t'/\tau\right)\int_0^t\int_0^{t'} Exp\left(u/\tau+v/\tau\right)E\left(\psi(u)\psi(v)\right)dudv
\end{aligned}
\tag{C.3}
$$

We have used that the covariance of the initial value ξ (0) and the noise is zero, to read $E\left(\xi(0)\psi(u)\right)=E\left(\xi(0)\right)E\left(\psi(u)\right)=0$. The integral in (C.3), which is quite general, can be calculated explicitly for a time dependent uncorrelated noise where $E\left(\psi(t)\psi(t')\right)=\xi_0^2\lambda\delta(t-t')$, to read

$$
\begin{aligned}
&Cov\left[\xi(t),\xi(t')\right]-Exp\left(-t/\tau-t'/\tau\right)Var\left[\xi(0)\right] \\
&=\xi_0^2\lambda Exp\left(-t/\tau-t'/\tau\right)\int_0^{t'}\int_0^t Exp\left(u/\tau+v/\tau\right)\delta\left(u-v\right)dudv \\
&=\xi_0^2\lambda Exp\left(-t/\tau-t'/\tau\right)\int_0^{t'}\int_{-v}^{-v+t} Exp\left(2v/\tau\right)Exp\left(z/\tau\right)\delta\left(z\right)dzdv \\
&=\xi_0^2\lambda Exp\left(-t/\tau-t'/\tau\right)\int_0^{t'} Exp\left(2v/\tau\right)dv=\frac{\xi_0^2\lambda}{2a}Exp\left(-t/\tau-t'/\tau\right)\left(Exp\left(2t'/\tau\right)-1\right) \\
&\approx\frac{\xi_0^2\lambda\tau}{2}Exp\left(-t/\tau+t'/\tau\right),t'/\tau\gg 1,t>t'
\end{aligned}
\tag{C.4}
$$

We choose $\lambda=1/\tau^2$.

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