Advanced Studies in Theoretical Physics
Vol. 10, 2016, no. 3, 113 - 124
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/astp.2016.512110

Mathematical Relations Related to the Lode Parameter for Studies of Ductility

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Abstract

During plastic flow, hydrostatic pressure and plastic strain control the size of the yield surface while the Lode angle is responsible for its shape. The von Mises yield condition shows no dependency of the Lode angle while the Tresca condition shows. It has been concluded that the deviation from the von Mises criterion is real and could not be explained on the basis of lack of experimental accuracy and isotropy. What is notable is that from an engineering viewpoint the accuracy of the von Mises yielding is amply sufficient. However, it has been forecast that Lode dependency must be included to describe ductility. It has been shown that internal necking down of ligaments between voids that have become significantly enlarged in size dominates at high triaxiality, while internal shear localization of plastic strain between voids of limited growth dominates at low stress triaxiality. The Lode angle has been used to account for these effects. In this paper, various mathematical relations are examined that may be useful for further studies of the Lode angle and its relation to fracture and ductility.

Keywords: Lode angel, ductility, fracture, triaxiality

1 Introduction

A plastic theory is constructed from a function of stress which defines the combinations of stress for which plastic yield takes place. The next component is a flow rule, which defines the ratio of the strain components as a function of the stress state at yield. This can often be put in terms of a normal to the plastic potential. In many cases the yield function can be used as the plastic potential (associated flow). The classical metal plasticity J_2 theory assumes that hydrostatic pressure has no or negligible effect on the material strain hardening and that the flow stress is independent of the third stress invariant (or the Lode angle parameter). In general, the hydrostatic pressure and the plastic strain control the size of the yield surface while the Lode (1926) angle is responsible for its shape. The von Mises (1913) yield condition shows no dependency of the Lode (1926) angle, while the Tresca (1864) condition shows Lode dependency. In 1931 Taylor and Quinney published their classical experiment for copper and steel tubes which was intended to settle the question related to use of Tresca or Mises criterion for plastic flow. They concluded that the deviation from the von Mises criterion was real and could not be explained on the basis of lack of experimental accuracy and isotropy. However, the data fit the von Mises criterion considerably better than the Tresca criterion. Attempts have been made over the years to improve the correlation of data by including the effect for the third stress invariant J_3 into the yield criterion. It has been shown that from an engineering viewpoint the accuracy of the von Mises yielding is amply sufficient.

The fracture process of ductile materials is known to be caused by nucleation of void, void growth and finally coalescence of voids to fracture. The fracture coalescence depends on triaxiality (that means on I_1 and J_2) (Van Stone et al. 1985, Garrison Jr. W.M). However, on the macroscopic level it has now been questioned whether triaxiality can fully describe many instances of isotropic ductility. It has been forecast that Lode dependency must be included to describe isotropic ductility. It has been shown that internal necking down of ligaments between voids that have enlarged their sizes significantly, dominates at high triaxiality, while internal shear localization of plastic strain ligaments between voids that have experienced limited growth dominates at low stress triaxiality (Parodoen and Brechet 2004). The Lode angle has been used to account for these effects.

To quantify the influence of stress triaxiality on ductility, different experiments on smoothed and notched bars are traditionally utilized (Hancock and Mackenzie 1976). In general, the larger the triaxiality, the smaller the fracture strains at failure. This is in agreement with theoretical models for void growth (McClintock 1968, Rice and Tracey 1969). However, McClintock (1971) and Johnson-Cook (1985) find that for many materials, the plastic strain to fracture was smaller in torsion (no triaxiality) compared to tension (larger triaxiality). Bao and Wierzbicki (2004 ab) recently compared different models to examine the influence of triaxiality. They concluded that none of the existing models were able to capture the behavior in the entire triaxiality range. For large triaxialities (say above 0.4), void growth was the dominating failure mode, while at low triaxialities shear of voids dominates. The main conclusion was that there is a possible slope discontinuity in the fracture locus corresponding to the point of fracture transition (Bao and Wierzbicki 2004b). The influence of the Lode angle parameter on fracture, void growth and coalescence has been investigated by recent studies (Zhang et al. 2001, Kim et al. 2004, Gao and Kim 2006, Wierzbicki et al. 2005, Barsoum and Faleskog 2007, Gruben et al. 2012).

In this article we study various mathematical relations that may be useful for further studies of the Lode parameter.

2 The different Lode angle parameters

For isotropic materials, the azimuth angle of a hydrostatic plane can be divided into six regions. In each sextant, the azimuth angle can be characterized by the Lode angle. The principal stress decomposition given by Bai and Wierzbicki (2010) is in the plastic π-plane given by (for definitions of stress invariants see Appendix A):

$$s_1 = \frac{2}{3}\sigma_{eq}Cos(\theta), s_2 = \frac{2}{3}\sigma_{eq}Cos\left(\frac{2}{3}\pi - \theta\right), s_3 = \frac{2}{3}\sigma_{eq}Cos\left(\frac{4}{3}\pi - \theta\right) \tag{2.1}$$

θ is the Lode angle, σ_{eq} is the von Mises equivalent stress and $s_i, i = 1,2,3$ are the principal stress deviators. We easily verify that the decomposition is mathematically valid, to read

$$\begin{aligned} s_1 + s_2 + s_3 &= \frac{2}{3}\sigma_{eq}\left(Cos(\theta) + Cos\left(\frac{2}{3}\pi - \theta\right) + Cos\left(\frac{4}{3}\pi - \theta\right)\right) \\ &= \frac{2}{3}\sigma_{eq}\left(Cos(\theta) + Cos\left(\frac{2}{3}\pi\right)Cos(\theta) + Sin\left(\frac{2}{3}\pi\right)Sin(\theta) + Cos\left(\frac{4}{3}\pi\right)Cos(\theta) + Sin\left(\frac{4}{3}\pi\right)Sin(\theta)\right) \\ &= \frac{2}{3}\sigma_{eq}\left(Cos(\theta) - \frac{1}{2}Cos(\theta) + \frac{\sqrt{3}}{2}Sin(\theta) - \frac{1}{2}Cos(\theta) - \frac{\sqrt{3}}{2}Sin(\theta)\right) = 0 \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \frac{3}{2}\left(s_1^2 + s_2^2 + s_3^2\right) &= \frac{2}{3}\sigma_{eq}{}^2\left(Cos(\theta)^2 + Cos\left(\frac{2}{3}\pi - \theta\right)^2 + Cos\left(\frac{4}{3}\pi - \theta\right)^2\right) \\ &= \frac{2}{3}\sigma_{eq}{}^2\left(Cos(\theta)^2 + \left(-\frac{1}{2}Cos(\theta) + \frac{\sqrt{3}}{2}Sin(\theta)\right)^2 + \left(-\frac{1}{2}Cos(\theta) - \frac{\sqrt{3}}{2}Sin(\theta)\right)^2\right) \\ &= \frac{2}{3}\sigma_{eq}{}^2\left(Cos(\theta)^2 + \frac{1}{4}Cos(\theta)^2 + \frac{3}{4}Sin(\theta)^2 + \frac{1}{4}Cos(\theta)^2 + \frac{3}{4}Sin(\theta)^2\right) = \sigma_{eq}{}^2 \end{aligned} \tag{2.3}$$

This further gives the principal stresses as:

$$\begin{aligned}\sigma_1 &= s_1 + \sigma_m = \left(1 + \frac{2}{3\sigma^*} Cos(\theta)\right)\sigma_m \\ \sigma_2 &= s_2 + \sigma_m = \left(1 + \frac{2}{3\sigma^*} Cos\left(\frac{2}{3}\pi - \theta\right)\right)\sigma_m \\ \sigma_3 &= s_3 + \sigma_m = \left(1 + \frac{2}{3\sigma^*} Cos\left(\frac{4}{3}\pi - \theta\right)\right)\sigma_m\end{aligned} \tag{2.4}$$

where $\sigma_m = 1/3(\sigma_1 + \sigma_2 + \sigma_3)$ and $\sigma^* = \sigma_m / \sigma_{eq}$ is the triaxiality. We define (see Appendix A) and find:

$$\begin{aligned}\xi &\overset{def}{=} \frac{r^3}{\sigma_{eq}^3} \frac{\left(\frac{27}{2} s_1 s_2 s_3\right)}{\sigma_{eq}^3} = 4Cos(\theta) Cos\left(\frac{2}{3}\pi - \theta\right) Cos\left(\frac{4}{3}\pi - \theta\right) = Cos(3\theta) \\ \theta &= \frac{1}{3} ArcCos(\xi) = \frac{1}{3} ArcCos\left(\frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}}\right)\end{aligned} \tag{2.5}$$

Notable is that $-1 \le \xi \le 1$ when $\theta \in [0, \pi/3]$. We now define the first Lode angle parameter by

$$\begin{aligned}\bar{\theta}_1 &\overset{def}{=} 1 - \frac{2}{\pi} ArcCos(\xi_1) = 1 - \frac{2}{\pi} ArcCos(Cos(3\theta)) = 1 - \frac{6}{\pi}\theta \\ &\Rightarrow \theta = \pi/6 - \bar{\theta}_1 \pi/6\end{aligned} \tag{2.6}$$

Thus the Lode angle relates linearly to the first Lode angle parameter. Moreover we define the second Lode angle parameter (opposite of Lode's definition) by

$$\begin{aligned}\mu = \bar{\theta}_2 &\overset{def}{=} \frac{2\sigma_2 - \sigma_1 - \sigma_3}{\sigma_1 - \sigma_3} = \frac{2s_2 - s_1 - s_3}{s_1 - s_3} = \frac{2Cos\left(\frac{2}{3}\pi - \theta\right) - Cos(\theta) - Cos\left(\frac{4}{3}\pi - \theta\right)}{Cos(\theta) - Cos\left(\frac{4}{3}\pi - \theta\right)} \\ &= \frac{-1 + \sqrt{3}Tan(\theta)}{1 + 1/\sqrt{3}Tan(\theta)} \Rightarrow Tan(\theta) = \frac{1 + \mu}{\sqrt{3} - \mu/\sqrt{3}}\end{aligned} \tag{2.7}$$

This second Lode angle parameter can be made more like the first Lode angle parameter if we change the phase (thus changing sign (Lode 1926)), to read:

$$\begin{aligned} &\bar{\theta}_3 \overset{def}{=} \bar{\theta}_2(\pi/3-\theta) = \frac{1-\sqrt{3}\,Tan(\theta)}{1+1/\sqrt{3}\,Tan(\theta)} = -\bar{\theta}_2(\theta) = -\mu \\ &\Rightarrow Tan(\theta) = \frac{1-\bar{\theta}_3(\theta)}{\sqrt{3}+\bar{\theta}_3(\theta)/\sqrt{3}} \end{aligned} \tag{2.8}$$

We notice in Figure 2.1 that $\bar{\theta}_3 = -\mu$ is indeed a very good approximation of $\bar{\theta}_1(\theta)$.

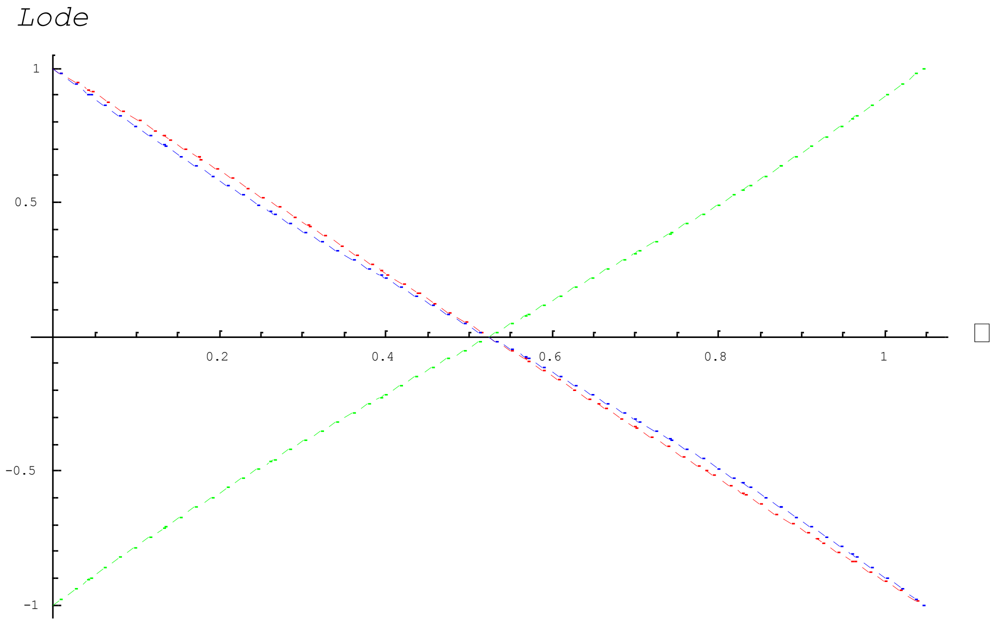

Figure 2.1: Lode angle parameters as a function of the Lode angle
Red: $\bar{\theta}_1$, Green: $\bar{\theta}_2$, Blue: $\bar{\theta}_3$=-$\bar{\theta}_2$=- μ

Instead of the parameter set $(\sigma_1, \sigma_2, \sigma_3)$ to characterize the stress space, we can use a cylindrical set $(\sigma_m, \sigma_{eq}, \theta) \rightarrow (z, \rho, \theta)$. We can also use the coordinate set $(\sigma_{eq}, \sigma^*, \theta)$.

This can be associated with the spherical coordinate set $(\sigma_{eq}, \sigma^*, \theta) \rightarrow (r, \varphi, \theta)$, where

$$Tan(\varphi) = \sqrt{2/3}\,\sigma_{eq} /(\sigma_m \sqrt{3}) = \frac{\sqrt{2}}{3\sigma^*}.$$

3 More on the Lode angle parameters

An isotropic model can be explained by the three stress invariants or by the pressure, von Mises stress and the third stress invariant. The Lode parameter associates to the third stress invariant. Equation (2.4) can be written as

$$\sigma_1 = \left(\sigma^* + \frac{2}{3}Cos(\theta)\right)\sigma_{eq}, \sigma_2 = \left(\sigma^* + \frac{2}{3}Cos\left(\frac{2}{3}\pi - \theta\right)\right)\sigma_{eq}, \sigma_3 = \left(\sigma^* + \frac{2}{3}Cos\left(\frac{4}{3}\pi - \theta\right)\right)\sigma_{eq} \tag{3.1}$$

Further inserting the first Lode angle parameter $\theta = \pi/6 - \bar{\theta}_1 \pi/6$ gives

$$\begin{aligned} \sigma_1 &= \left(\sigma^* + \frac{2}{3}Cos\left(\pi/6 - \bar{\theta}_1 \pi/6\right)\right)\sigma_{eq}, \\ \sigma_2 &= \left(\sigma^* + \frac{2}{3}Cos\left(\frac{3}{6}\pi + \bar{\theta}_1 \pi/6\right)\right)\sigma_{eq}, \\ \sigma_3 &= \left(\sigma^* + \frac{2}{3}Cos\left(\frac{7}{6}\pi + \bar{\theta}_1 \pi/6\right)\right)\sigma_{eq} \end{aligned} \tag{3.2}$$

This gives the principal stresses as a function of the triaxiality, von Mises equivalent stress and the first Lode angle parameter. Using the second Lode angle parameter gives
$\theta = ArcTan\left((1+\mu)/(\sqrt{3} - \mu/\sqrt{3})\right)$. Thus equation (3.2) becomes

$$\begin{aligned} \sigma_1 &= \left(\sigma^* + \frac{2}{3}Cos(\theta)\right)\sigma_{eq} = \left(\sigma^* + \frac{2}{3}Cos\left(ArcTan\left(\frac{1+\mu}{\sqrt{3} - \mu/\sqrt{3}}\right)\right)\right)\sigma_{eq} = \left(\sigma^* + \frac{3-\mu}{3\sqrt{3+\mu^2}}\right)\sigma_{eq} \\ \sigma_2 &= \left(\sigma^* + \frac{2}{3}Cos\left(\frac{2}{3}\pi - \theta\right)\right)\sigma_{eq} = \left(\sigma^* + \frac{2\mu}{3\sqrt{3+\mu^2}}\right)\sigma_{eq} \\ \sigma_3 &= \left(\sigma^* + \frac{2}{3}Cos\left(\frac{4}{3}\pi - \theta\right)\right)\sigma_{eq} = \left(\sigma^* - \frac{3+\mu}{3\sqrt{3+\mu^2}}\right)\sigma_{eq} \end{aligned} \tag{3.3}$$

This is in agreement with Gruben et al. (2012). Using the Lode angle we achieve that

$$\frac{\sigma_1 - \sigma_3}{\sigma_{eq}} = \frac{2}{3}Cos(\theta) - \frac{2}{3}Cos\left(\frac{4}{3}\pi - \theta\right) = Cos(\theta) + 1/\sqrt{3}Sin(\theta) \tag{3.4}$$

Using the first Lode angle parameter we achieve from equation (3.4)

$$\frac{\sigma_1-\sigma_3}{\sigma_{eq}}=Cos\left(\pi/6-\bar{\theta}_1\pi/6\right)+1/\sqrt{3}Sin\left(\pi/6-\bar{\theta}_1\pi/6\right)=\frac{2}{\sqrt{3}}Cos\left(\bar{\theta}_1\pi/6\right) \tag{3.5}$$

Using the second Lode angle parameter equation (3.3) gives

$$\frac{\sigma_1-\sigma_3}{\sigma_{eq}}=\frac{3-\mu}{3\sqrt{3+\mu^2}}+\frac{3+\mu}{3\sqrt{3+\mu^2}}=\frac{2}{\sqrt{3+\mu^2}} \tag{3.6}$$

Lets us assume that $\sigma_1 \geq \sigma_2 \geq \sigma_3$. The Tresca and von Mises criterion gives

$$\begin{aligned}&Tresca:\frac{\sigma_1-\sigma_3}{\sigma_{eq}}=1\\&Mises:\frac{\sigma_1-\sigma_3}{\sigma_{eq}}=\frac{2}{\sqrt{3+\mu^2}}=\frac{1}{\sqrt{3}}\frac{2}{\sqrt{1+\left(\mu/\sqrt{3}\right)^2}}\end{aligned} \tag{3.7}$$

Thus only for $\mu=\pm 1$ are the two relations alike. This corresponds to $\theta=\left\{0,\pi/3\right\}\Rightarrow\bar{\theta}_1=\mp 1$.

We define the second Lode angle by

$$Tan\left(\theta_2\right)=\mu/\sqrt{3} \tag{3.8}$$

This gives that

$$\begin{aligned}&Tan\left(\theta\right)=\frac{1+\sqrt{3}Tan\left(\theta_2\right)}{\sqrt{3}-Tan\left(\theta_2\right)}\Rightarrow Tan\left(\theta\right)\sqrt{3}Tan\left(\theta_2\right)=1+\sqrt{3}Tan\left(\theta_2\right)\\&\Rightarrow Tan\left(\theta_2\right)=\frac{1}{\left(Tan\left(\theta\right)-1\right)\sqrt{3}}\end{aligned} \tag{3.9}$$

During a plane stress situation, $\sigma_3=0$. This gives that

$$\sigma_3 = \left(\sigma^* + \frac{2}{3}Cos\left(\frac{4}{3}\pi - \theta\right)\right)\sigma_{eq} = \left(\sigma^* - \frac{3+\mu}{3\sqrt{3+\mu^2}}\right)\sigma_{eq} = 0$$

$$\mu^2\left(9\sigma^{*2}-1\right) - 6\mu + 27\sigma^{*2} - 9 = 0 \Rightarrow \mu = \frac{3 \pm 3\sqrt{1-\left(9\sigma^{*2}-1\right)\left(3\sigma^{*2}-1\right)}}{\left(9\sigma^{*2}-1\right)} \tag{3.10}$$

$$\theta = \frac{4}{3}\pi - ArcCos\left(-\frac{3}{2}\sigma^*\right), \theta_1 = 1 - \frac{6}{\pi}\theta$$

We also have that

$$\xi = Cos(3\theta) = Cos\left(4\pi - 3ArcCos\left(-\frac{3}{2}\sigma^*\right)\right) = -\frac{27}{2}\sigma^*\left(\sigma^{*2} - \frac{1}{3}\right) \tag{3.11}$$

For the plane stress we can write that

$$\sigma_3 = 0, \sigma_2 = \alpha\,\sigma_1, \sigma_m = \frac{1}{3}\left(\sigma_1 + \sigma_2\right) = \frac{1}{3}\left(1+\alpha\right)\sigma_1$$
$$s_1 = \sigma_1 - \sigma_m = \sigma_1 - \frac{1}{3}\left(1+\alpha\right)\sigma_1 = \frac{1}{3}\left(2-\alpha\right)\sigma_1, \tag{3.12}$$
$$s_2 = \sigma_2 - \sigma_m = \alpha\,\sigma_1 - \frac{1}{3}\left(1+\alpha\right)\sigma_1 = -\frac{1}{3}\sigma_1 + \frac{2}{3}\alpha\sigma_1 = \frac{1}{3}\left(2\alpha-1\right)\sigma_1$$

This gives that

$$\mu = \frac{2\sigma_2 - \sigma_1}{\sigma_1} = 2\alpha - 1$$
$$\sigma^* = \frac{3+\mu}{3\sqrt{3+\mu^2}} = \frac{3+\left(2\alpha-1\right)}{3\sqrt{3+\left(2\alpha-1\right)^2}} = \frac{1+\alpha}{3\sqrt{1+\alpha^2-\alpha}} \tag{3.13}$$

We generally write that

$$\varepsilon_1^p \geq 0, \varepsilon_2^p = \beta\varepsilon_1^p, \varepsilon_z^p = -\varepsilon_1^p - \beta\varepsilon_1^p = -\left(1+\beta\right)\varepsilon_1^p \tag{3.14}$$

The plastic equations give that

$$d\varepsilon_{ij}^p = \frac{s_{ij}}{2k}d\lambda, \beta = d\varepsilon_2^p / d\varepsilon_1^p = s_2 / s_1 = \frac{2\alpha-1}{2-\alpha} \Rightarrow \alpha = \frac{2\beta+1}{\beta+2} \tag{3.15}$$

Neglecting elastic components give that $d\varepsilon_{ij}^{p} = \varepsilon_{ij}^{p}$. This gives that

$$\sigma^{*} = \frac{1+\alpha}{3\sqrt{1+\alpha^{2}-\alpha}} = \frac{1+\left(\frac{2\beta+1}{\beta+2}\right)}{3\sqrt{1+\left(\frac{2\beta+1}{\beta+2}\right)^{2}-\left(\frac{2\beta+1}{\beta+2}\right)}} = \frac{\sqrt{3}}{3}\frac{\beta+1}{\sqrt{1+\beta^{2}+\beta}} \tag{3.16}$$

$$\mu = 2\alpha - 1 = \frac{4\beta+2}{\beta+2} - 1 = \frac{4\beta+2-\beta-2}{\beta+2} = \frac{3\beta}{\beta+2}$$

4 Conclusion

It has been forecast that Lode dependency must be included to describe ductility. It has been shown that internal necking down of ligaments between voids that have become significantly enlarged in size dominates at high triaxiality, while internal shear localization of plastic strain ligaments between voids that have experienced limited growth dominates at low stress triaxiality. We study various mathematical relations that may be useful for further studies of the Lode angle and its relation to fracture and ductility.

Appendix A: Stress deviators and Lode angle parameters

The stress space can be identified by the three principal components $\sigma_1, \sigma_2, \sigma_3$ or by the invariants of the stress tensor, to read

$$\begin{aligned}
I_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_m \\
I_2 &= \sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2 - \left(\sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz}\right) = -\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right) \\
I_3 &= \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{xz}\sigma_{yz} - \left(\sigma_{xx}\sigma_{yz}^2 + \sigma_{yy}\sigma_{xz}^2 + \sigma_{zz}\sigma_{xy}^2\right) = \sigma_1\sigma_2\sigma_3
\end{aligned} \tag{A.1}$$

The invariants of the stress deviator $s_{ij} = \sigma_{ij} - \sigma_m\delta_{ij}, \sigma_m = \frac{1}{3}\left(\sigma_{ii}\right)$ are

$$\begin{aligned}
J_1 &= 0 \\
J_2 &= \frac{1}{3}\left(I_1^2 + 3I_2\right) = \frac{1}{6}\left[\left(\sigma_{xx} - \sigma_{yy}\right)^2 + \left(\sigma_{xx} - \sigma_{zz}\right)^2 + \left(\sigma_{yy} - \sigma_{zz}\right)^2 + 6\left(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2\right)\right] = \frac{1}{2}s_{ij}{}^2 \\
&= \frac{1}{6}\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_1 - \sigma_3\right)^2 + \left(\sigma_2 - \sigma_3\right)^2\right] = \frac{1}{2}\left(s_1{}^2 + s_2{}^2 + s_3{}^2\right) = -\left(s_1s_2 + s_1s_3 + s_2s_3\right) \\
J_3 &= \frac{1}{27}\left(2I_1^3 + 9I_1I_2 + 27I_3\right) = Det\left(s_{ij}\right) = s_1s_2s_3 = \frac{1}{3}\left(s_1{}^3 + s_2{}^3 + s_3{}^3\right)
\end{aligned}$$

(A.2)

It is also common to define

$$\sigma_{eq} \overset{def}{=} \sqrt{3J_2} = \sqrt{I_1^2 + 3I_2} = \sqrt{\frac{3}{2}\left(s_1{}^2 + s_2{}^2 + s_3{}^2\right)}$$

$$r \overset{def}{=} \left(\frac{27}{2}J_3\right)^{1/3} = \left(\frac{1}{2}\left(2I_1^3 + 9I_1I_2 + 27I_3\right)\right)^{1/3} = \left(\frac{27}{2}s_1s_2s_3\right)^{1/3}$$

$$\sigma^* \overset{def}{=} \frac{\sigma_m}{\sigma_{eq}} = \frac{1}{3}\left(\frac{I_1}{I_1^2 + 3I_2}\right)$$

$$\xi \overset{def}{=} \left(\frac{r}{\sigma_{eq}}\right)^3 = \frac{2I_1^3 + 9I_1I_2 + 27I_3}{2\left(I_1^2 + 3I_2\right)^3} = \frac{\left(\frac{27}{2}s_1s_2s_3\right)}{\left(\frac{3}{2}\left(s_1{}^2 + s_2{}^2 + s_3{}^2\right)\right)^{3/2}}$$

$$Tresca : 4J_2^3 - 27J_2^2 - 36k^2J_2^2 + 96k^4J_2 - 64k^6 = 0$$

$$(\varepsilon^p\ )^2 = \frac{2}{9}\left[\left(\varepsilon_x^p - \varepsilon_y^p\right)^2 + \left(\varepsilon_x^p - \varepsilon_z^p\right)^2 + \left(\varepsilon_y^p - \varepsilon_z^p\right)^2 + 6\varepsilon_{xy}^p + 6\varepsilon_{xz}^p + 6\varepsilon_{yz}^p\right] = \frac{2}{3}\left(\varepsilon_1^{p2} + \varepsilon_2^{p2} + \varepsilon_3^{p2}\right) \tag{A.3}$$

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Received: December 15, 2015; Published: March 11, 2016